

THE BOOTSTRAP MULTISCALE ANALYSIS FOR THE MULTI-PARTICLE ANDERSON MODEL

ABEL KLEIN AND SON T. NGUYEN

ABSTRACT. We extend the bootstrap multi-scale analysis developed by Germinet and Klein to the multi-particle Anderson model, obtaining Anderson localization, dynamical localization, and decay of eigenfunction correlations.

CONTENTS

1. Introduction	1
2. Preliminaries to the multiscale analysis	5
2.1. Partially and fully separated boxes	5
2.2. Wegner estimates	6
2.3. Partially and fully interactive boxes	7
2.4. Resonant boxes	9
2.5. Suitable Cover	9
3. The bootstrap multiscale analysis	11
3.1. The first multiscale analysis	12
3.2. The second multiscale analysis	15
3.3. The third multiscale analysis	18
3.4. The fourth multiscale analysis	20
3.5. Completing the proof of the bootstrap multiscale analysis	27
4. Localization for the multi-particle Anderson model	27
4.1. Anderson Localization	28
4.2. Dynamical Localization	29
4.3. SUDEC	32
References	36

1. INTRODUCTION

Localization is by now well understood for the Anderson model, a random Schrödinger operator that describes an electron moving in a medium with random impurities (e.g., the review [Ki]). More recently, localization has been proved for a multi-particle Anderson model with a finite range interaction potential by Chulaevsky and Suhov [CS1, CS2, CS3] and Aizenman and Warzel [AW]. Chulaevsky and Suhov used a multiscale analysis based on [DK] and Aizenman and Warzel [AW] employed the fractional moment method as in [ASFH]. Chulaevsky, Boutet de Monvel, and Suhov [CBS] extended the results of Chulaevsky and Suhov to the continuum multi-particle Anderson model.

A.K. was supported in part by the NSF under grant DMS-1001509.

In this article we extend the bootstrap multi-scale analysis developed by Germinet and Klein [GK1, Kl] to the multi-particle Anderson model, obtaining Anderson localization, dynamical localization, and decay of eigenfunction correlations. The advantage of our method is that it extends to the continuum multi-particle Anderson model, yielding the strong localization results proven in [GK1, GK2, Kl] for the one particle continuum Anderson model. This extension will appear in a sequel to this paper.

We start by defining the n -particle Anderson model.

Definition 1.1. *The n -particle Anderson model is the random Schrödinger operator on $\ell^2(\mathbb{Z}^{nd})$ given by*

$$H_{\omega}^{(n)} := -\Delta^{(n)} + V_{\omega}^{(n)} + U, \quad (1.1)$$

where:

- (i) $\Delta^{(n)}$ is the discrete nd -dimensional Laplacian operator.
- (ii) $\omega = \{\omega_x\}_{x \in \mathbb{Z}^d}$ is a family of independent identically distributed random variables whose common probability distribution μ has a bounded density ρ with compact support.
- (iii) $V_{\omega}^{(n)}$ is the random potential given by

$$V_{\omega}^{(n)}(\mathbf{x}) = \sum_{i=1, \dots, n} V_{\omega}^{(1)}(x_i), \quad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^{nd}, \quad (1.2)$$

where $V_{\omega}^{(1)}(x) = \omega_x$ for every $x \in \mathbb{Z}^d$.

- (iv) U is a potential governing the short range interaction between the n particles. We take

$$U(\mathbf{x}) = \sum_{1 \leq i < j \leq n} \tilde{U}(x_i - x_j), \quad (1.3)$$

where $\tilde{U}: \mathbb{Z}^d \rightarrow \mathbb{R}$, $\tilde{U}(y) = \tilde{U}(-y)$, and $\tilde{U}(y) = 0$ for $\|y\|_{\infty} > r_0$ for some $0 < r_0 < \infty$.

Remark. We took a two-body interaction potential in (1.3) for simplicity, but our results would still be valid with a more general finite range interaction potential as in [AW].

We will generally omit ω from the notation, and use the following notation:

- (i) Given $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we set $\|x\| = \|x\|_{\infty} := \max\{|x_1|, \dots, |x_d|\}$. If $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^{nd}$, we let $\|\mathbf{a}\| := \max\{\|a_1\|, \dots, \|a_n\|\}$, $\langle \mathbf{a} \rangle := \sqrt{1 + \|\mathbf{a}\|^2}$, and $\mathcal{S}_{\mathbf{a}} = \{a_1, \dots, a_n\}$.
- (ii) Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{nd}$, we set $d_H(\mathbf{a}, \mathbf{b}) := d_H(\mathcal{S}_{\mathbf{a}}, \mathcal{S}_{\mathbf{b}})$, where $d_H(S_1, S_2)$ denotes the Hausdorff distance between finite subsets $S_1, S_2 \subseteq \mathbb{R}^d$, given by

$$\begin{aligned} d_H(S_1, S_2) &:= \max \left\{ \max_{x \in S_1} \min_{y \in S_2} \|x - y\|, \max_{y \in S_2} \min_{x \in S_1} \|x - y\| \right\} \\ &= \max \left\{ \max_{x \in S_1} \text{dist}(x, S_2), \max_{y \in S_2} \text{dist}(y, S_1) \right\}. \end{aligned} \quad (1.4)$$

- (iii) We use n -particle boxes in \mathbb{Z}^{nd} centered at points in \mathbb{R}^{nd} . The n -particle box of side $L \geq 1$ centered at $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^{nd}$ is given by

$$\Lambda_L^{(n)}(\mathbf{x}) = \{\mathbf{y} \in \mathbb{Z}^{nd}; \|\mathbf{y} - \mathbf{x}\| \leq \frac{L}{2}\} = \prod_{i=1}^n \Lambda_L(x_i) \subseteq \mathbb{Z}^{nd}. \quad (1.5)$$

By a box Λ_L in \mathbb{Z}^{nd} we mean an n -particle box $\Lambda_L^{(n)}(\mathbf{x})$. Note that

$$(L-2)^{nd} < |\Lambda_L^{(n)}(\mathbf{x})| \leq (L+1)^{nd}. \quad (1.6)$$

Since we always work with L large, we will use $|\Lambda_L^{(n)}(\mathbf{x})| \leq L^{nd}$ and ignore the small error.

- (iv) We will occasionally use boxes in \mathbb{R}^{nd} . We set

$$\widehat{\Lambda}_L^{(n)}(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^{nd}; \|\mathbf{y} - \mathbf{x}\| \leq \frac{L}{2}\}; \quad \text{note} \quad \Lambda_L^{(n)}(\mathbf{x}) = \widehat{\Lambda}_L^{(n)} \cap \mathbb{Z}^{nd}. \quad (1.7)$$

- (v) Given a box $\Lambda_t^{(n)} \subseteq \Lambda_L^{(n)} = \Lambda_L^{(n)}(\mathbf{x})$, we let

$$\partial \Lambda_L^{(n)} \Lambda_t^{(n)} = \left\{ (\mathbf{u}, \mathbf{v}) \in \Lambda_t^{(n)} \times (\Lambda_L^{(n)} \setminus \Lambda_t^{(n)}) \mid \|\mathbf{u} - \mathbf{v}\|_1 = 1 \right\}, \quad (1.8)$$

$$\partial_+ \Lambda_L^{(n)} \Lambda_t^{(n)} = \left\{ \mathbf{v} \in \Lambda_L^{(n)} \setminus \Lambda_t^{(n)} \mid (\mathbf{u}, \mathbf{v}) \in \partial \Lambda_t^{(n)} \text{ for some } \mathbf{u} \in \Lambda_t^{(n)} \right\}.$$

Note that there exists a constant s_{Nd} such that for $t \geq 1$ we have

$$\left| \partial_+ \Lambda_L^{(n)} \Lambda_t^{(n)} \right| \leq \left| \partial \Lambda_L^{(n)} \Lambda_t^{(n)} \right| \leq s_{Nd} t^{nd}. \quad (1.9)$$

When it is clear that $\Lambda_t^{(n)} \subseteq \Lambda_L^{(n)}$ we will simply write $\partial \Lambda_t^{(n)}$ and $\partial_+ \Lambda_t^{(n)}$.

- (vi) Given an n -particle box $\Lambda = \Lambda_L^{(n)}(\mathbf{x})$, we define the finite volume operator $H_\Lambda = H_{\Lambda_L^{(n)}(\mathbf{x})}^{(n)}$ as the self-adjoint operator on $\ell^2(\Lambda)$ obtained by restricting $H^{(n)}$ to Λ with Dirichlet (simple) boundary condition: $H_\Lambda = \chi_\Lambda H^{(n)} \chi_\Lambda$ restricted to $\ell^2(\Lambda)$. If $z \notin \sigma(H_\Lambda)$, we set

$$G_\Lambda(z) = (H_\Lambda - z)^{-1}, \quad G_\Lambda(z; \mathbf{u}, \mathbf{y}) = \langle \delta_{\mathbf{u}}, (H_\Lambda - z)^{-1} \delta_{\mathbf{y}} \rangle \text{ for } \mathbf{u}, \mathbf{y} \in \Lambda. \quad (1.10)$$

We will use several types of good boxes. Note that they are defined for a fixed ω (omitted from the notation).

Definition 1.2. Let $\Lambda = \Lambda_L^{(n)}(\mathbf{x})$ be an n -particle box and let $E \in \mathbb{R}$. Then:

- (i) Given $\theta > 0$, the n -particle box Λ is said to be (θ, E) -suitable if, and only if, $E \notin \sigma(H_\Lambda)$ and

$$|G_\Lambda(E; \mathbf{a}, \mathbf{b})| \leq L^{-\theta} \text{ for all } \mathbf{a}, \mathbf{b} \in \Lambda \text{ with } \|\mathbf{a} - \mathbf{b}\| \geq \frac{L}{100}. \quad (1.11)$$

Otherwise, Λ is called (θ, E) -unsuitable.

- (ii) Given $\zeta \in (0, 1)$, the n -particle box Λ is said to be (ζ, E) -subexponentially suitable (SES) if, and only if, $E \notin \sigma(H_\Lambda)$ and

$$|G_\Lambda(E; \mathbf{a}, \mathbf{b})| \leq e^{-L^\zeta} \text{ for all } \mathbf{a}, \mathbf{b} \in \Lambda \text{ with } \|\mathbf{a} - \mathbf{b}\| \geq \frac{L}{100}. \quad (1.12)$$

Otherwise, $\Lambda_L^{(n)}(\mathbf{x})$ is called (ζ, E) -nonsubexponentially suitable (nonSES).

- (iii) Given $m > 0$, the n -particle box Λ is said to be (m, E) -regular if, and only if, $E \notin \sigma(H_\Lambda)$ and

$$|G_\Lambda(E; \mathbf{a}, \mathbf{b})| \leq e^{-m\|\mathbf{a}-\mathbf{b}\|} \quad \text{for all } \mathbf{a}, \mathbf{b} \in \Lambda \quad \text{with} \quad \|\mathbf{a} - \mathbf{b}\| \geq \frac{L}{100}. \quad (1.13)$$

Otherwise, Λ is called (m, E) -nonregular.

Remark 1.3. It follows immediately from the definitions that:

- (i) $\Lambda_L^{(n)}(\mathbf{x})$ (m^*, E) -regular $\implies \Lambda_L^{(n)}(\mathbf{x})$ $\left(\frac{m^*L}{100 \log L}, E\right)$ -suitable.
- (ii) $\Lambda_L^{(n)}(\mathbf{x})$ (θ, E) -suitable $\implies \Lambda_L^{(n)}(\mathbf{x})$ $\left(\frac{\theta \log L}{L}, E\right)$ -regular.
- (iii) $\Lambda_L^{(n)}(\mathbf{x})$ $(L^{\zeta-1}, E)$ -regular $\implies \Lambda_L^{(n)}(\mathbf{x})$ $\left(\zeta - \frac{\log 100}{\log L}, E\right)$ -SES.
- (iv) $\Lambda_L^{(n)}(\mathbf{x})$ (ζ, E) -SES $\implies \Lambda_L^{(n)}(\mathbf{x})$ $(L^{\zeta-1}, E)$ -regular.

We are ready to state our main result, which extends the bootstrap multiscale analysis of Germinet and Klein [GK1] to the multi-particle Anderson model with short range interaction.

Theorem 1.4. There exist $p_0(n) = p_0(d, n) > 0$, $n = 1, 2, \dots$, with the property that for every $N \in \mathbb{N}$, given $\theta > 8Nd$, there exists $\mathcal{L} = \mathcal{L}(d, \|\rho\|_\infty, N, \theta)$ such that if for some $L_0 \geq \mathcal{L}$ we have

$$\sup_{\mathbf{x} \in \mathbb{R}^{nd}} \mathbb{P} \left\{ \Lambda_{L_0}^{(n)}(\mathbf{x}) \text{ is } (\theta, E)\text{-nonsuitable} \right\} \leq p_0(n), \quad (1.14)$$

for every $E \in \mathbb{R}$ and every $n = 1, 2, \dots, N$, then, given $0 < \zeta < 1$, we can find a length scale $L_\zeta = L_\zeta(d, \|\rho\|_\infty, N, \theta, L_0)$, $\delta_\zeta = \delta_\zeta(d, \|\rho\|_\infty, N, \theta, L_0) > 0$, and $m_\zeta = m_\zeta(\delta_\zeta, L_\zeta) > 0$, so that the following holds for $n = 1, 2, \dots, N$:

- (i) For every $E \in \mathbb{R}$, $L \geq L_\zeta$, and $\mathbf{a} \in \mathbb{R}^{nd}$, we have

$$\mathbb{P} \left\{ \Lambda_L^{(n)}(\mathbf{a}) \text{ is } (m_\zeta, E)\text{-nonregular} \right\} \leq e^{-L^\zeta}. \quad (1.15)$$

- (ii) Given $E_1 \in \mathbb{R}$, set $I(E_1) = [E_1 - \delta_\zeta, E_1 + \delta_\zeta]$. Then, for every $E_1 \in \mathbb{R}$, $L \geq L_\zeta$, and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{nd}$ with $d_H(\mathbf{a}, \mathbf{b}) \geq L$, we have

$$\mathbb{P} \left\{ \exists E \in I(E_1) \text{ so } \Lambda_L^{(n)}(\mathbf{a}) \text{ and } \Lambda_L^{(n)}(\mathbf{b}) \text{ are } (m_\zeta, E)\text{-nonregular} \right\} \leq e^{-L^\zeta}. \quad (1.16)$$

Remark 1.5. The hypotheses of Theorem 1.4 are verified at high disorder. Consider the n -particle Anderson model given in Definition 1.1 with a disorder parameter $\lambda > 0$ (cf. (1.1)):

$$H_{\omega, \lambda}^{(n)} := -\Delta^{(n)} + \lambda V_{\omega}^{(n)} + U. \quad (1.17)$$

$H_{\omega, \lambda}$ can be rewritten as an n -particle Anderson model $H_{\omega}^{(\lambda)}$ in the exact form of Definition 1.1 by replacing the probability distribution μ by the probability distribution $\mu^{(\lambda)}$, defined by $\mu^{(\lambda)}(B) = \mu(\lambda^{-1}B)$ for all Borel sets $B \subset \mathbb{R}$, with density $\rho^{(\lambda)}(t) = \frac{1}{\lambda} \rho(\frac{t}{\lambda})$. Proceeding as in [DK, Proposition 3.1.2], we can show that for all $N \in \mathbb{N}$, given a scale L_0 , there exists $\lambda_N < \infty$, such that for all $\lambda \geq \lambda_N$ the condition (1.14) is satisfied at scale L_0 by $H_{\omega, \lambda}^{(n)}$ for every $E \in \mathbb{R}$ and every $n = 1, 2, \dots, N$. Since $\|\rho^{(\lambda)}\|_\infty = \frac{1}{\lambda} \|\rho\|_\infty \leq \|\rho\|_\infty$ for $\lambda \geq 1$, and $\|\rho^{(\lambda)}\|_\infty$ is the only constant that appears in the proof of the theorem that changes with λ , it follows that the conclusions of Theorem 1.4 are valid for all $\lambda \geq \lambda_N$ with the same constants $L_\zeta, \delta_\zeta, m_\zeta$.

Theorem 1.4 is proved in Section 3. The theorem is proved by induction on the number of particles. The one particle case was proven in [GK1, KI]. (These papers deal with the continuum Anderson model, but the results apply to the discrete Anderson model.) The proof of the induction step proceeds as in [GK1, KI], with four multi-scale analyses, using some technical arguments of [GK3]. To deal with the fact that in the multi-particle case events based on disjoint boxes are not independent, we use the partially and fully separated boxes and partially and fully interactive boxes introduced by Chulaevsky and Suhov [CS1, CS2, CS3]. The relevant distance between boxes is the Hausdorff distance, introduced in this context by Aizenman and Warzel [AW]. We prove a Wegner estimate (Theorem 2.3) and a Wegner estimate between partially separated boxes (Theorem 2.4). In the multiscale analysis partially interactive boxes are handled by the induction hypothesis, i.e., by the conclusions of Theorem 1.4 for a smaller number of particles (see Lemma 2.8), and fully interactive boxes are handled similarly to one particle boxes (see Lemma 2.10).

Theorem 1.4 implies localization: Anderson localization, dynamical localization, and estimates on the behavior of eigenfunctions.

Corollary 1.6. *Assume the conclusions of Theorem 1.4. Then:*

- (i) (Anderson localization) $H_{\omega}^{(N)}$ has pure point spectrum with exponentially decaying eigenfunctions for \mathbb{P} -a.e. ω .
- (ii) (Dynamical Localization) For every $\mathbf{y} \in \mathbb{Z}^{Nd}$ we can find a constant $C(\mathbf{y})$ such that

$$\mathbb{E} \left\{ \sup_{t \in \mathbb{R}} \left| \left\langle \delta_{\mathbf{x}}, e^{-itH_{\omega}^{(N)}} \delta_{\mathbf{y}} \right\rangle \right| \right\} \leq C(\mathbf{y}) e^{-d_H(\mathbf{x}, \mathbf{y})} \quad \text{for every } \mathbf{x} \in \mathbb{Z}^{Nd}. \quad (1.18)$$

- (iii) (Summable Uniform Decay of Eigenfunction Correlations (SUDEC)) Fix $\nu > \frac{Nd}{2} + \frac{1}{2}$ and let T be the operator on \mathcal{H} given by multiplication by the function $\langle \mathbf{x} \rangle^{2\nu}$. Then, for \mathbb{P} -a.e. ω H_{ω} has pure point spectrum in the open interval I with finite multiplicity, and for every $\zeta \in (0, 1)$ there exists a constant $C_{\omega, \zeta}$ such that for every eigenvalue E of $H_{\omega}^{(N)}$ and $\psi, \phi \in \text{Ran } \chi_{\{E\}}(H_{\omega})$, we have that, for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}$,

$$|\phi(\mathbf{x})| |\psi(\mathbf{y})| \leq C_{\omega, \zeta} \|T^{-1}\phi\| \|T^{-1}\psi\| \langle \mathbf{x} \rangle^{2\nu} e^{-d_H(\mathbf{x}-\mathbf{y})^{\zeta}}, \quad \text{and} \quad (1.19)$$

$$|\phi(\mathbf{x})| |\psi(\mathbf{y})| \leq C_{\omega, \zeta} \|T^{-1}\phi\| \|T^{-1}\psi\| \langle \mathbf{x} \rangle^{\nu} \langle \mathbf{y} \rangle^{\nu} e^{-d_H(\mathbf{x}-\mathbf{y})^{\zeta}}. \quad (1.20)$$

Corollary 1.6 is proven in Section 4.

2. PRELIMINARIES TO THE MULTISCALE ANALYSIS

2.1. Partially and fully separated boxes. We call $\Lambda^{(n)}(\mathbf{a}) = \prod_{i=1}^n \Lambda_{L_i}(a_i)$ an n -particle rectangle centered at $\mathbf{a} \in \mathbb{R}^{nd}$. Given subsets $\mathcal{J}, \mathcal{K} \subseteq \{1, \dots, n\}$, with $\mathcal{K} \neq \emptyset$, we set

$$\Pi_i \Lambda^{(n)}(\mathbf{a}) = \Lambda_L(a_i), \quad \Pi_{\mathcal{J}} \Lambda^{(n)}(\mathbf{a}) = \bigcup_{i \in \mathcal{J}} \Lambda_L(a_i), \quad \Pi \Lambda^{(n)}(\mathbf{a}) = \Pi_{\{1, \dots, n\}} \Lambda^{(n)}(\mathbf{a}),$$

$$\mathbf{a}_{\mathcal{K}} = (a_i, i \in \mathcal{K}), \quad \mathbf{a} = (\mathbf{a}_{\mathcal{K}}, \mathbf{a}_{\mathcal{K}^c}), \quad \Lambda(\mathbf{a}_{\mathcal{K}}) = \Lambda^{\mathcal{K}}(\mathbf{a}_{\mathcal{K}}) = \prod_{i \in \mathcal{K}} \Lambda_{L_i}(a_i).$$

Definition 2.1. Let $\Lambda^{(n)}(\mathbf{x}) = \prod_{i=1}^n \Lambda_{L_i}(x_i)$ and $\Lambda^{(n)}(\mathbf{y}) = \prod_{i=1}^n \Lambda_{L_i}(y_i)$ be a pair of n -particle rectangles.

- (i) $\Lambda^{(n)}(\mathbf{x})$ and $\Lambda^{(n)}(\mathbf{y})$ are partially separated if, and only if, either $\Lambda_{L_i}(x_i) \cap \Pi\Lambda^{(n)}(\mathbf{y}) = \emptyset$ for some $i \in \{1, \dots, n\}$, or $\Lambda_{\ell_j}(y_j) \cap \Pi\Lambda^{(n)}(\mathbf{x}) = \emptyset$ for some $j \in \{1, \dots, n\}$.
- (ii) $\Lambda^{(n)}(\mathbf{x})$ and $\Lambda^{(n)}(\mathbf{y})$ are fully separated if, and only if,

$$\left(\Pi\Lambda^{(n)}(\mathbf{x})\right) \cap \left(\Pi\Lambda^{(n)}(\mathbf{y})\right) = \emptyset. \quad (2.1)$$

Given a pair of n -particle rectangles $\Lambda^{(n)}(\mathbf{x})$ and $\Lambda^{(n)}(\mathbf{y})$ as above, with $L_i, \ell_i \leq L$ for all $i \in \{1, \dots, n\}$, if there exists $i \in \{1, \dots, n\}$ such that $\|x_i - y_j\| \geq L$ for every $j \in \{1, \dots, n\}$, then $\Lambda_{L_i}(x_i) \cap \Pi\Lambda^{(n)}(\mathbf{y}) = \emptyset$. In other words, if there exists $i \in \{1, \dots, n\}$ such that $\text{dist}(x_i, \mathcal{S}_{\mathbf{y}}) \geq L$, then $\Lambda_{L_i}(x_i) \cap \Pi\Lambda^{(n)}(\mathbf{y}) = \emptyset$. We have the following lemma.

Lemma 2.2. *Let $\Lambda^{(n)}(\mathbf{x}) = \prod_{i=1}^n \Lambda_{L_i}(x_i) \subseteq \Lambda_L^{(n)}(\mathbf{x})$ and $\Lambda^{(n)}(\mathbf{y}) = \prod_{i=1}^n \Lambda_{\ell_i}(y_i) \subseteq \Lambda_L^{(n)}(\mathbf{y})$ be a pair of n -particle rectangles. Then*

- (i) $\Lambda^{(n)}(\mathbf{x})$ and $\Lambda^{(n)}(\mathbf{y})$ are partially separated if $d_H(\mathbf{x}, \mathbf{y}) > L$.
- (ii) $\Lambda_L^{(n)}(\mathbf{x})$ and $\Lambda_L^{(n)}(\mathbf{y})$ are fully separated if $\text{dist}(\mathcal{S}_{\mathbf{x}}, \mathcal{S}_{\mathbf{y}}) > L$.

2.2. Wegner estimates. Wegner estimates have been previously proved for the n -particle Anderson model (e.g., [CS1, CS3]). We derive optimal Wegner estimates, that is, with the expected dependence on the volume and interval length.

Theorem 2.3. *Consider the n -particle rectangle $\Lambda = \prod_{i=1, \dots, n} \Lambda_{L_i}(a_i) \subseteq \Lambda_L^{(n)}(\mathbf{a})$ and let $\Gamma = \Lambda_{L_k}(a_k)$ for some $k \in \{1, \dots, n\}$. Then for any interval I we have*

$$\mathbb{E}_{\Gamma} \left(\text{tr } \chi_I(H_{\omega, \Lambda}) \right) \leq n \|\rho\|_{\infty} |I| L^{nd}. \quad (2.2)$$

In particular, for any $E \in \mathbb{R}$ and $\varepsilon > 0$ we have

$$\mathbb{P}_{\Gamma} \left\{ \|G_{\Lambda}(E)\| \geq \frac{1}{\varepsilon} \right\} = \mathbb{P}_{\Gamma} \left\{ d(\sigma(H_{\Lambda}), E) \leq \varepsilon \right\} \leq 2n \|\rho\|_{\infty} \varepsilon L^{nd}. \quad (2.3)$$

Proof. We begin by rewriting $H_{\omega, \Lambda}^{(n)}$ as

$$H_{\omega, \Lambda}^{(n)} = -\Delta_{\Lambda} + U_{\Lambda} + \sum_{\mathbf{x} \in \Lambda} \sum_{i=1, \dots, n} \omega_{x_i} \Pi_{\mathbf{x}}, \quad (2.4)$$

where $\Pi_{\mathbf{x}}$ denotes the rank one orthogonal projection onto $\delta_{\mathbf{x}}$. Given $y \in \mathbb{Z}^d$, we set $q_y(\mathbf{x}) = \#\{i = 1, \dots, n \mid x_i = y\}$ for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^{nd}$. Then (see [AW])

$$\begin{aligned} H_{\omega, \Lambda}^{(n)} &= -\Delta_{\Lambda} + U_{\Lambda} + \sum_{\mathbf{x} \in \Lambda} \sum_{y \in \mathbb{Z}^d} q_y(\mathbf{x}) \omega_y \Pi_{\mathbf{x}} = -\Delta_{\Lambda} + U_{\Lambda} + \sum_{y \in \mathbb{Z}^d} \omega_y \sum_{\mathbf{x} \in \Lambda} q_y(\mathbf{x}) \Pi_{\mathbf{x}} \\ &= -\Delta_{\Lambda} + U_{\Lambda} + \sum_{y \in \mathbb{Z}^d} \omega_y \Theta_y = -\Delta_{\Lambda} + U_{\Lambda} + \sum_{y \notin \Gamma} \omega_y \Theta_y + \sum_{y \in \Gamma} \omega_y \Theta_y, \end{aligned} \quad (2.5)$$

where $\Theta_y = \sum_{\mathbf{x} \in \Lambda} q_y(\mathbf{x}) \Pi_{\mathbf{x}}$.

Let I be an interval. Given $\tilde{y} \in \Gamma$, we set

$$\tilde{H}_{(\omega_{\tilde{y}})^{\perp}, \Lambda} = H_{\omega, \Lambda}^{(n)} - \omega_{\tilde{y}} \Theta_{\tilde{y}} = -\Delta_{\Lambda} + U_{\Lambda} + \sum_{y \notin \Gamma} \omega_y \Theta_y + \sum_{y \in \Gamma \setminus \{\tilde{y}\}} \omega_y \Theta_y. \quad (2.6)$$

As functions on \mathbb{Z}^{nd} , we have $\chi_{\Lambda} \leq \sum_{y \in \Gamma} \Theta_y$, so

$$\mathrm{tr} \chi_I(H_{\omega}, \Lambda) \leq \sum_{y \in \Gamma} \mathrm{tr} \left(\Theta_y \chi_I(H_{\omega}, \Lambda) \right) = \sum_{y \in \Gamma} \mathrm{tr} \left(\Theta_y \chi_I(\tilde{H}_{(\omega_y)^\perp, \Lambda} + \omega_y \Theta_y) \right). \quad (2.7)$$

Hence,

$$\begin{aligned} \mathbb{E}_{\Gamma}(\mathrm{tr} \chi_I(H_{\omega}, \Lambda)) &\leq \sum_{y \in \Gamma} \mathbb{E}_{\omega_y^\perp} \mathbb{E}_{\omega_y} \left\{ \mathrm{tr} \sqrt{\Theta_y} \chi_I(\tilde{H}_{(\omega_y)^\perp, \Lambda} + \omega_y \Theta_y) \sqrt{\Theta_y} \right\} \\ &\leq \sum_{y \in \Gamma} \mathbb{E}_{\omega_y^\perp} \left\{ \|\rho\|_\infty |I| n L^{(n-1)d} \right\} \leq |\Gamma| \|\rho\|_\infty |I| n L^{(n-1)d}, \end{aligned} \quad (2.8)$$

since $\dim \mathrm{Ran} \Theta_y \leq n L^{(n-1)d}$. But $|\Gamma| \leq L^d$, so we conclude that

$$\mathbb{E}_{\Gamma}(\mathrm{tr} \chi_I(H_{\omega}, \Lambda)) \leq n \|\rho\|_\infty |I| L^{nd}. \quad (2.9)$$

□

Corollary 2.4. *Let $\Lambda_1 = \prod_{i=1, \dots, n} \Lambda_{L_i}(a_i) \subseteq \Lambda_L^{(n)}(\mathbf{a})$ and $\Lambda_2 = \prod_{i=1, \dots, n} \Lambda_{L'_i}(b_i) \subseteq \Lambda_L^{(n)}(\mathbf{a})$ be a pair of partially separated n -particle rectangles. Then*

$$\mathbb{P} \left\{ d \left(\sigma(H_{\Lambda_1}), \sigma(H_{\Lambda_2}) \right) \leq \varepsilon \right\} \leq 2n \|\rho\|_\infty \varepsilon L^{2nd} \quad \text{for all } \varepsilon > 0. \quad (2.10)$$

Proof. Let Λ_1, Λ_2 be as above. Since they are partially separated there is, $\Gamma = \Lambda_{L_k}(a_k)$, such that $\Gamma \cap \Pi \Lambda_2 = \emptyset$. Note that H_{Λ_2} depends only on ω_{Γ^c} , and thus $\sigma(H_{\Lambda_2}) = \left\{ E_1(\omega_{\Gamma^c}), \dots, E_{|\Lambda_2|}(\omega_{\Gamma^c}) \right\}$, where $E_j(\omega_{\Gamma^c})$ is independent of ω_{Γ} . Thus

$$\begin{aligned} \mathbb{P} \left\{ d \left(\sigma(H_{\Lambda_1}), \sigma(H_{\Lambda_2}) \right) \leq \varepsilon \right\} &= \mathbb{E}_{\Gamma^c} \mathbb{P}_{\Gamma} \left\{ d \left(\sigma(H_{\Lambda_1}), \sigma(H_{\Lambda_2}) \right) \leq \varepsilon \right\} \\ &= \mathbb{E}_{\Gamma^c} \mathbb{P}_{\Gamma} \left\{ d \left(\sigma(H_{\Lambda_1}), E_j(\omega_{\Gamma^c}) \right) \leq \varepsilon \text{ for some } j = 1, \dots, |\Lambda_2| \right\} \\ &\leq \mathbb{E}_{\Gamma^c} \sum_{j=1, \dots, |\Lambda_2|} \mathbb{P}_{\Gamma} \left\{ d \left(\sigma(H_{\Lambda_1}), E_j(\omega_{\Gamma^c}) \right) \leq \varepsilon \right\} \leq |\Lambda_2| (2n \|\rho\|_\infty \varepsilon L^{nd}), \end{aligned} \quad (2.11)$$

using (2.3). The estimate (2.10) follows since $|\Lambda_2| \leq L^{nd}$. □

2.3. Partially and fully interactive boxes. Following Chulaevsky and Suhov [CS2, CS3], we divide boxes into partially and fully interactive.

Definition 2.5. *An n -particle box $\Lambda_L^{(n)}(\mathbf{a})$ is said to be partially interactive (PI) if and only if there exists a nonempty proper subset $\mathcal{J} \subsetneq \{1, \dots, n\}$ such that*

$$\Lambda_L^{(n)}(\mathbf{a}) \subseteq \mathcal{E}_{\mathcal{J}}, \quad \text{where } \mathcal{E}_{\mathcal{J}} = \left\{ \mathbf{x} \in \mathbb{Z}^{nd} \mid \min_{i \in \mathcal{J}, j \notin \mathcal{J}} \|x_i - x_j\| > r_0 \right\}. \quad (2.12)$$

If $\Lambda_L^{(n)}(\mathbf{a})$ is not partially interactive, it is said to be fully interactive (FI).

Remark 2.6. *If the n -particle box $\Lambda_L^{(n)}(\mathbf{u})$ is partially interactive, by writing $\Lambda_L^{(n)}(\mathbf{u}) = \Lambda_L^{\mathcal{J}}(\mathbf{u}_{\mathcal{J}}) \times \Lambda_L^{\mathcal{J}^c}(\mathbf{u}_{\mathcal{J}^c})$ we are implicitly stating that $\Lambda_L^{(n)}(\mathbf{u}) \subseteq \mathcal{E}_{\mathcal{J}}$ for some nonempty proper subset $\mathcal{J} \subsetneq \{1, \dots, n\}$. Moreover, we set $\sigma_{\mathcal{J}} = \sigma \left(H_{\Lambda_L^{\mathcal{J}}(\mathbf{u}_{\mathcal{J}})} \right)$ and $\sigma_{\mathcal{J}^c} = \sigma \left(H_{\Lambda_L^{\mathcal{J}^c}(\mathbf{u}_{\mathcal{J}^c})} \right)$.*

Lemma 2.7. *Let $\Lambda_L^{(n)}(\mathbf{u}) = \Lambda_L^{\mathcal{J}}(\mathbf{u}_{\mathcal{J}}) \times \Lambda_L^{\mathcal{J}^c}(\mathbf{u}_{\mathcal{J}^c})$ be a PI n -particle box. Then:*

$$(i) \quad \Pi_{\mathcal{J}} \Lambda_L^{(n)}(\mathbf{u}) \cap \Pi_{\mathcal{J}^c} \Lambda_L^{(n)}(\mathbf{u}) = \emptyset.$$

- (ii) $H_{\Lambda_L^{(n)}(\mathbf{a})} = H_{\Lambda_L^{\mathcal{J}}(\mathbf{u}_{\mathcal{J}})} \otimes I_{\Lambda_L^{\mathcal{J}^c}(\mathbf{u}_{\mathcal{J}^c})} + I_{\Lambda_L^{\mathcal{J}}(\mathbf{u}_{\mathcal{J}})} \otimes H_{\Lambda_L^{\mathcal{J}^c}(\mathbf{u}_{\mathcal{J}^c})}$.
 (iii) For all $E \notin \sigma(\Lambda_L^{(n)}(\mathbf{u}))$ and $\mathbf{a}, \mathbf{b} \in \Lambda_L^{(n)}(\mathbf{u})$ we have

$$\left| G_{\Lambda_L^{(n)}(\mathbf{u})}(E; \mathbf{a}, \mathbf{b}) \right| \leq \sum_{\lambda \in \sigma_{\mathcal{J}}} \left| G_{\Lambda_L(\mathbf{u}_{\mathcal{J}^c})}(E - \lambda; \mathbf{a}_{\mathcal{J}^c}, \mathbf{b}_{\mathcal{J}^c}) \right|, \quad (2.13)$$

$$\left| G_{\Lambda_L^{(n)}(\mathbf{u})}(E; \mathbf{a}, \mathbf{b}) \right| \leq \sum_{\mu \in \sigma_{\mathcal{J}^c}} \left| G_{\Lambda_L(\mathbf{u}_{\mathcal{J}})}(E - \mu; \mathbf{a}_{\mathcal{J}}, \mathbf{b}_{\mathcal{J}}) \right|. \quad (2.14)$$

Proof. (i) and (ii) follows from the definition of a PI box. To prove (iii), given $\lambda \in \sigma_{\mathcal{J}}$ we let $\Pi_{\lambda}^{(\mathcal{J})}$ denote the orthogonal projection onto the corresponding eigenspace. If $E \notin \sigma(\Lambda_L^{(n)}(\mathbf{u}))$, it follows from (ii) that

$$G_{\Lambda_L^{(n)}(\mathbf{u})}(E) = \sum_{\lambda \in \sigma_{\mathcal{J}}, \mu \in \sigma_{\mathcal{J}^c}} (E - \lambda - \mu)^{-1} \Pi_{\lambda}^{(\mathcal{J})} \otimes \Pi_{\mu}^{(\mathcal{J}^c)}, \quad (2.15)$$

which implies (2.13) and (2.14). \square

As a consequence, we get

Lemma 2.8. Let $\Lambda_{\ell}^{(n)}(\mathbf{u}) = \Lambda_{\ell}^{\mathcal{J}}(\mathbf{u}_{\mathcal{J}}) \times \Lambda_{\ell}^{\mathcal{J}^c}(\mathbf{u}_{\mathcal{J}^c})$ be a PI n -particle box and $E \in \mathbb{R}$. If ℓ is sufficiently large, the following holds:

- (i) Given $\theta > 0$, suppose $\Lambda_{\ell}(\mathbf{u}_{\mathcal{J}})$ is $(\theta, E - \mu)$ -suitable for every $\mu \in \sigma_{\mathcal{J}^c}$ and $\Lambda_{\ell}(\mathbf{u}_{\mathcal{J}^c})$ is $(\theta, E - \lambda)$ -suitable for every $\lambda \in \sigma_{\mathcal{J}}$. Then $\Lambda_{\ell}^{(n)}(\mathbf{u})$ is $(\frac{\theta}{2}, E)$ -suitable.
- (ii) Given $m > 0$, suppose $\Lambda_{\ell}(\mathbf{u}_{\mathcal{J}})$ is $(m, E - \mu)$ -regular for every $\mu \in \sigma_{\mathcal{J}^c}$ and $\Lambda_{\ell}(\mathbf{u}_{\mathcal{J}^c})$ is $(m, E - \lambda)$ -regular for every $\lambda \in \sigma_{\mathcal{J}}$. Then $\Lambda_{\ell}^{(n)}(\mathbf{u})$ is $(m - \frac{100nd \log \ell}{\ell}, E)$ -regular.
- (iii) Given $0 < \zeta' < \zeta < 1$, suppose $\Lambda_{\ell}(\mathbf{u}_{\mathcal{J}})$ is $(\zeta, E - \mu)$ -SES for every $\mu \in \sigma_{\mathcal{J}^c}$ and $\Lambda_{\ell}(\mathbf{u}_{\mathcal{J}^c})$ is $(\zeta, E - \lambda)$ -SES for every $\lambda \in \sigma_{\mathcal{J}}$. Then $\Lambda_{\ell}^{(n)}(\mathbf{u})$ is (ζ', E) -SES.

Proof. We prove (i), the proofs of (ii) and (ii) are similar. Given $\mathbf{a}, \mathbf{b} \in \Lambda_{\ell}^{(N)}(\mathbf{u})$ with $\|\mathbf{a} - \mathbf{b}\| \geq \frac{\ell}{100}$, then either we have $\|\mathbf{a}_{\mathcal{J}} - \mathbf{b}_{\mathcal{J}}\| \geq \frac{\ell}{100}$, or $\|\mathbf{a}_{\mathcal{J}^c} - \mathbf{b}_{\mathcal{J}^c}\| \geq \frac{\ell}{100}$. Without loss of generality, we suppose that $\|\mathbf{a}_{\mathcal{J}} - \mathbf{b}_{\mathcal{J}}\| \geq \frac{\ell}{100}$. Then

$$\left| G_{\Lambda_{\ell}^{(N)}(\mathbf{u})}(E; \mathbf{a}, \mathbf{b}) \right| \leq \sum_{\mu \in \sigma_{\mathcal{J}^c}} \left| G_{\Lambda_{\ell}(\mathbf{u}_{\mathcal{J}})}(E - \mu; \mathbf{a}_{\mathcal{J}}, \mathbf{b}_{\mathcal{J}}) \right| \quad (2.16)$$

$$\leq |\Lambda_{\ell}(\mathbf{u}_{\mathcal{J}^c})| \ell^{-\theta} \leq \ell^{nd-\theta} \leq \ell^{-\frac{\theta}{2}}, \quad (2.17)$$

provided ℓ is sufficiently large. \square

Definition 2.9. Let $\Lambda_L^{(n)}(\mathbf{a})$ and $\Lambda_L^{(n)}(\mathbf{b})$ be a pair of n -particle boxes. We say that $\Lambda_L^{(n)}(\mathbf{a})$ and $\Lambda_L^{(n)}(\mathbf{b})$ are L -distant if $\max\{\text{dist}(\mathbf{b}, \mathcal{S}_{\mathbf{a}}^n), \text{dist}(\mathbf{a}, \mathcal{S}_{\mathbf{b}}^n)\} \geq 2nL$.

For fully interactive boxes we have the following lemma.

Lemma 2.10. Let $\Lambda_L^{(n)}(\mathbf{a})$ and $\Lambda_L^{(n)}(\mathbf{b})$ be a pair of FI n -particle boxes, where $L \geq 2(n-1)r_0$. Then $\Lambda_L^{(n)}(\mathbf{a})$ and $\Lambda_L^{(n)}(\mathbf{b})$ are fully separated if

$$\max_{x \in \mathcal{S}_{\mathbf{a}}, y \in \mathcal{S}_{\mathbf{b}}} \|x - y\| \geq 2nL. \quad (2.18)$$

In particular, L -distant FI n -particle boxes are fully separated.

Proof. If a box $\Lambda_L^{(n)}(\mathbf{a})$ is FI, we have $\max_{x, x' \in \mathcal{S}_\mathbf{a}} \|x - x'\| \leq (n-1)(L+r_0)$. Thus, if $\Lambda_L^{(n)}(\mathbf{a})$ and $\Lambda_L^{(n)}(\mathbf{b})$ are FI, and $L \geq 2(n-1)r_0$, the condition (2.18) implies

$$\text{dist}(\mathcal{S}_\mathbf{a}, \mathcal{S}_\mathbf{b}) = \min_{x \in \mathcal{S}_\mathbf{a}, y \in \mathcal{S}_\mathbf{b}} \|x - y\| \geq 2nL - 2(n-1)(L+r_0) > L, \quad (2.19)$$

so $\Lambda_L^{(n)}(\mathbf{a})$ and $\Lambda_L^{(n)}(\mathbf{b})$ are fully separated.

Since

$$\max\{\text{dist}(\mathbf{x}, \mathcal{S}_\mathbf{y}^n), \text{dist}(\mathbf{y}, \mathcal{S}_\mathbf{x}^n)\} \leq \max_{x \in \mathcal{S}_\mathbf{a}, y \in \mathcal{S}_\mathbf{b}} \|x - y\|, \quad (2.20)$$

we conclude that L -distant FI n -particle boxes are fully separated. \square

2.4. Resonant boxes.

Definition 2.11. Let $\Lambda = \prod_{i=1, \dots, n} \Lambda_{L_i}(a_i)$ be an n -particle box, $L = \min_{i=1, \dots, n} \{L_i\}$, and $E \in \mathbb{R}$.

- (i) Let $s > 0$. Then Λ is called (E, s) -suitably resonant if and only if $\text{dist}(\sigma(H_\Lambda^{(n)}), E) < L^{-s}$. Otherwise, Λ is said to be (E, s) -suitably nonresonant.
- (ii) Let $\beta \in (0, 1)$. Then Λ is called (E, β) -resonant if and only if $\text{dist}(\sigma(H_\Lambda^{(n)}), E) < \frac{1}{2}e^{-L^\beta}$. Otherwise, Λ is said to be (E, β) -nonresonant.

2.5. Suitable Cover. We now introduce the concept of a suitable cover as in [GK3, Definition 3.12], adapted to the discrete case.

Definition 2.12. Let $\Lambda_L^{(n)}(\mathbf{x})$ be an n -particle box, and $\ell < L$. The suitable ℓ -covering of $\Lambda_L^{(n)}(\mathbf{x})$ is the collection of n -particle boxes

$$\mathcal{C}_{L, \ell}^{(n)}(\mathbf{x}) = \{\Lambda_\ell^{(n)}(\mathbf{a})\}_{\mathbf{a} \in \Xi_{L, \ell}^{(n)}}, \quad (2.21)$$

where

$$\Xi_{L, \ell}^{(n)} := \{\mathbf{x} + \alpha \ell \mathbb{Z}^{nd}\} \cap \widehat{\Lambda}_L^{(n)} \quad \text{with} \quad \alpha = \max\left[\frac{3}{5}, \frac{4}{5}\right] \cap \left\{\frac{L-\ell}{2\ell k}; k \in \mathbb{N}\right\}. \quad (2.22)$$

We recall [GK3, Lemma 3.13], which we rewrite in our context.

Lemma 2.13. Let $\ell \leq \frac{L}{6}$. Then for every n -particle box $\Lambda_L^{(n)}(\mathbf{x})$ the suitable ℓ -covering $\mathcal{C}_{L, \ell}^{(n)}(\mathbf{x})$ satisfies

$$\Lambda_L^{(n)}(\mathbf{x}) = \bigcup_{\mathbf{a} \in \Xi_{L, \ell}^{(n)}} \Lambda_\ell^{(n)}(\mathbf{a}), \quad (2.23)$$

$$\text{for } \mathbf{b} \in \Lambda_L^{(n)}(\mathbf{x}) \text{ there is } \Lambda_\ell^{(n, \mathbf{b})} \in \mathcal{C}_{L, \ell}^{(n)}(\mathbf{x}) \text{ with } \Lambda_\ell^{(n)}(\mathbf{b}) \cap \Lambda_L^{(n)}(\mathbf{x}) \subseteq \Lambda_\ell^{(n, \mathbf{b})}, \quad (2.24)$$

$$\Lambda_\ell^{(n)}(\mathbf{a}) \cap \Lambda_\ell^{(n)}(\mathbf{b}) = \emptyset \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathbf{x} + \alpha \ell \mathbb{Z}^{nd}, \mathbf{a} \neq \mathbf{b}, \quad (2.25)$$

$$\left(\frac{L}{\ell}\right)^{nd} \leq \#\Xi_{L, \ell}^{(n)} = \left(\frac{L-\ell}{\alpha \ell} + 1\right)^{nd} \leq \left(\frac{2L}{\ell}\right)^{nd}. \quad (2.26)$$

Moreover, given $\mathbf{a} \in \mathbf{x} + \alpha \ell \mathbb{Z}^{nd}$ and $k \in \mathbb{N}$, it follows that

$$\Lambda_{(2k\alpha+1)\ell}^{(n)}(\mathbf{a}) = \bigcup_{\mathbf{b} \in \{\mathbf{x} + \alpha \ell \mathbb{Z}^{nd}\} \cap \widehat{\Lambda}_{(2k\alpha+1)\ell}^{(n)}(\mathbf{b})} \Lambda_\ell^{(n)}(\mathbf{b}). \quad (2.27)$$

Note that $\Lambda_\ell^{(n,b)}$ does not denote a box centered at \mathbf{b} , just some box in $\mathcal{C}_{L,\ell}^{(n)}(\mathbf{x})$ satisfying (2.24). By $\Lambda_\ell^{(n,b)}$, or just $\Lambda_\ell^{(b)}$, we will always mean such a box.

Remark 2.14. *It suffices to require $\alpha \in [\frac{3}{5}, \frac{4}{5}] \cap \{\frac{L-\ell}{2\ell k}; k \in \mathbb{N}\}$ in Definition 2.12. We specified $\alpha = \alpha_{L,\ell}$ for convenience, so there is no ambiguity in the definition of $\mathcal{C}_{L,\ell}^{(n)}(\mathbf{x})$.*

Lemma 2.15. *Let $\Lambda_L^{(N)}(\mathbf{x})$ be an N -particle box and $\ell < \frac{L}{6}$. Define $\{k_j\}_{j \in \mathbb{N}} \subset \mathbb{N}$ by*

$$k_1 = 6 \text{ and } k_j = \min \left\{ k \in \mathbb{N} \mid k > k_{j-1} + 6 + 2(N\alpha)^{-1} \right\} \text{ for } j = 2, 3, \dots, \\ \text{so } k_j \leq 6 + (j-1) \left(7 + 2(N\alpha)^{-1} \right) \leq j \left(7 + 2(N\alpha)^{-1} \right), \quad (2.28)$$

and set

$$K_j = 2k_j N\alpha + 1 \leq 17jN \quad \text{for } j = 1, 2, \dots \quad (2.29)$$

Then, given $\mathbf{a}_s \in \Xi_{L,\ell}^{(N)}(\mathbf{x})$, $s = 1, 2, \dots, S$, we can find $\mathbf{u}_t \in \Xi_{L,\ell}^{(N)}(\mathbf{x})$ with $t = 1, 2, \dots, T \leq SN^N$, and $j_t \in \{1, 2, \dots, SN^N\}$ such that

$$\tilde{\Upsilon} = \tilde{\Upsilon}_{L,\ell}^{(N)}(\{\mathbf{a}_s\}_{s=1}^S) := \bigcup_{t=1}^T \Lambda_{K_{j_t}\ell}^{(N)}(\mathbf{u}_t) \subseteq \Lambda_L^{(N)}(\mathbf{x}), \quad (2.30)$$

$$\text{dist} \left\{ \Lambda_{K_{j_t}\ell}^{(N)}(\mathbf{u}_t), \Lambda_{K_{j_{t'}}\ell}^{(N)}(\mathbf{u}_{t'}) \right\} > 1 \quad \text{for } t \neq t', \quad (2.31)$$

$$\partial_+ \Lambda_{K_{j_t}\ell}^{(N)}(\mathbf{u}_t) \cap \tilde{\Upsilon} = \emptyset \quad \text{for } t = 1, 2, \dots, T, \quad (2.32)$$

$$\sum_{t=1}^T K_{j_t} \leq 17SN^{N+1} \quad (2.33)$$

and for $\mathbf{y} \in \Lambda_L^{(N)}(\mathbf{x}) \setminus \tilde{\Upsilon}$ and $\Lambda_\ell^{(N,\mathbf{y})} \in \mathcal{C}_{L,\ell}^{(N)}(\mathbf{x})$ as in (2.24), the boxes $\Lambda_\ell^{(N,\mathbf{y})}$ and $\Lambda_\ell(\mathbf{a}_s)$ are ℓ -distant for $s = 1, 2, \dots, S$.

Proof. Given $\mathbf{a} \in \Xi_{L,\ell}^{(n)}$, we set

$$\Upsilon_{L,\ell}^{(N)}(\mathbf{x}, \mathbf{a}) = \bigcup_{\mathbf{b} \in S_{\mathbf{a}}^N} \Lambda_{(4N+2)\ell}^{(N)}(\mathbf{b}) \cap \Lambda_L^{(N)}(\mathbf{x}), \quad (2.34)$$

and note that for $\mathbf{y} \in \Lambda_L^{(N)}(\mathbf{x}) \setminus \Upsilon_{L,\ell}^{(N)}(\mathbf{x}, \mathbf{a})$ and $\Lambda_\ell^{(N,\mathbf{y})} \in \mathcal{C}_{L,\ell}^{(N)}(\mathbf{x})$ as in (2.24), the boxes $\Lambda_\ell^{(N,\mathbf{y})}$ and $\Lambda_\ell(\mathbf{a})$ are ℓ -distant.

Given $\mathbf{a}_s \in \Xi_{L,\ell}^{(N)}(\mathbf{x})$, $s = 1, 2, \dots, S$, we set $\Upsilon = \bigcup_{s=1}^S \Upsilon_{L,\ell}^{(N)}(\mathbf{x}, \mathbf{a}_s)$. In view of (2.23) and (2.27), we can find $\mathbf{b}_r \in \Xi_{L,\ell}^{(N)}(\mathbf{x})$, $r = 1, 2, \dots, R \leq SN^N$, such that (we use $(12N\alpha + 1)\ell > (4N + 2)\ell + 2\alpha\ell$)

$$\Upsilon \subseteq \bigcup_{r=1}^R \Lambda_{(12N\alpha+1)\ell}^{(N)}(\mathbf{b}_r) \subseteq \Lambda_L^{(N)}(\mathbf{x}). \quad (2.35)$$

Let $\{k_j\}_{j \in \mathbb{N}} \subset \mathbb{N}$ be as in (2.28), so in particular

$$\frac{1}{2}(2k_j N\alpha + 1)\ell > \frac{1}{2}(2k_{j-1} N\alpha + 1)\ell + 1 + (2k_1 N\alpha + 1)\ell \quad \text{for } j = 2, 3, \dots \quad (2.36)$$

By a geometrical argument we can find $\mathbf{u}_t \in \Xi_{L,\ell}^{(N)}(\mathbf{x})$ and $j_t \in \{1, 2, \dots, SN^N\}$, $t = 1, 2, \dots, T \leq SN^N$, such that

$$\bigcup_{r=1}^R \Lambda_{(12N\alpha+1)\ell}^{(N)}(\mathbf{b}_r) \subseteq \tilde{\Upsilon} = \bigcup_{t=1}^T \Lambda_{K_{j_t}\ell}^{(N)}(\mathbf{u}_t) \subseteq \Lambda_L^{(N)}(\mathbf{x}), \quad (2.37)$$

and (2.31) holds, implying (2.32), and $\Lambda_{K_{j_t}\ell}^{(N)}(\mathbf{u}_t)$ contains at least j_t of the boxes $\Lambda_{(12N\alpha+1)\ell}^{(N)}(\mathbf{b}_r)$, so $\sum_{t=1}^T j_t \leq SN^N$. Thus, using (2.29),

$$\sum_{t=1}^T K_{j_t} \leq 17N \sum_{t=1}^T j_t \leq 17SN^{N+1}. \quad (2.38)$$

In view of (2.35) and (2.30), we conclude that for $\mathbf{y} \in \Lambda_L^{(N)}(\mathbf{x}) \setminus \tilde{\Upsilon}$ and $\Lambda_\ell^{(N,\mathbf{y})} \in \mathcal{C}_{L,\ell}^{(N)}(\mathbf{x})$ as in (2.24), the boxes $\Lambda_\ell^{(N,\mathbf{y})}$ and $\Lambda_\ell(\mathbf{a}_s)$ are ℓ -distant for $s = 1, 2, \dots, S$. \square

3. THE BOOTSTRAP MULTISCALE ANALYSIS

We will now prove Theorem 1.4 by induction on the number of particles, The one particle case was proven by Germinet and Klein [GK1]. We fix $N \geq 2$, assume Theorem 1.4 holds for $n = 1, 2, \dots, N-1$ particles, and prove the theorem for N particles. As in [GK1], the proof will be done by a bootstrapping argument, making successive use of four multiscale analyses.

In this section we assume that the following induction hypothesis; it follows from assuming that Theorem 1.4 holds for $n = 1, 2, \dots, N-1$.

Induction hypothesis. *For every $\tau \in (0, 1)$ there is a length scale L_τ , $\delta_\tau > 0$, and $m_\tau^* > 0$, such that the following hold for all $E \in \mathbb{R}$ and $n = 1, 2, \dots, N-1$:*

i) For all $L \geq L_\tau$ and $\mathbf{a} \in \mathbb{R}^{nd}$ we have

$$\mathbb{P}\left\{\Lambda_L^{(n)}(\mathbf{a}) \text{ is } (m_\tau^*, E)\text{-nonregular}\right\} \leq e^{-L^\tau}. \quad (3.1)$$

ii) Let $I(E) = [E - \delta_\tau, E + \delta_\tau]$. For all $L \geq L_\tau$ and all pairs of partially separated n -particle boxes $\Lambda_L^{(n)}(\mathbf{a})$ and $\Lambda_L^{(n)}(\mathbf{b})$ we have

$$\mathbb{P}\left\{\exists E' \in I(E) \text{ so both } \Lambda_L^{(n)}(\mathbf{a}) \text{ and } \Lambda_L^{(n)}(\mathbf{b}) \text{ are } (m_\tau^*, E')\text{-nonregular}\right\} \leq e^{-L^\tau}. \quad (3.2)$$

For partially interactive N -particle boxes we can estimate probabilities directly from the induction hypothesis, without a multiscale analysis for N -particles.

Lemma 3.1. *Let $\Lambda_\ell^{(N)}(\mathbf{u}) = \Lambda_\ell(\mathbf{u}_{\mathcal{J}}) \times \Lambda_\ell(\mathbf{u}_{\mathcal{J}^c})$ be a PI N -particle box and $\varsigma \in (0, 1)$. Then for ℓ large and all $E \in \mathbb{R}$ we have*

$$\begin{aligned} \mathbb{P}\left\{\Lambda_\ell^{(N)}(\mathbf{u}) \text{ is } (m_\varsigma^*(\ell), E)\text{-nonregular}\right\} &\leq \ell^{Nd} e^{-\ell^\varsigma} \text{ with } m_\varsigma^*(\ell) = m_\varsigma^* - \frac{100Nd \log \ell}{\ell}, \\ \mathbb{P}\left\{\Lambda_\ell^{(N)}(\mathbf{u}) \text{ is } (\theta, E)\text{-nonsuitable}\right\} &\leq \ell^{Nd} e^{-\ell^\varsigma} \text{ for } \theta < \frac{\ell}{\log \ell} \frac{m_\varsigma^*(\ell)}{100}, \\ \mathbb{P}\left\{\Lambda_\ell^{(N)}(\mathbf{u}) \text{ is } (\varsigma, E)\text{-nonSES}\right\} &\leq \ell^{Nd} e^{-\ell^\varsigma}. \end{aligned} \quad (3.3)$$

Proof. Let $E \in \mathbb{R}$, and ℓ large. It follows from Lemma 2.8(ii) and the induction hypothesis that

$$\begin{aligned} & \mathbb{P}\left\{\Lambda_\ell^{(N)}(\mathbf{u}) \text{ is } (m_\tau^*(\ell), E)\text{-nonregular}\right\} \\ & \leq \sum_{\mu \in \sigma_{\mathcal{J}^c}} \mathbb{P}\left\{\Lambda_\ell(\mathbf{u}_{\mathcal{J}}) \text{ is } (m_\tau^*, E - \mu,)\text{-nonregular}\right\} \\ & \quad + \sum_{\lambda \in \sigma_{\mathcal{J}}} \mathbb{P}\left\{\Lambda_\ell(\mathbf{u}_{\mathcal{J}^c}) \text{ is } (m_\tau^*, E - \lambda)\text{-nonregular}\right\} \leq \ell^{Nd} e^{-\ell^\tau}, \end{aligned} \quad (3.4)$$

where we used $\ell^{|\mathcal{J}|d} + \ell^{|\mathcal{J}^c|d} \leq \ell^{Nd}$ for ℓ large. The other estimates now follow from Remark 1.3. \square

In what follows, we fix $\zeta, \tau, \beta, \zeta_0, \zeta_1, \zeta_2, \gamma$ such that

$$\begin{aligned} 0 < \zeta < \tau < 1, \quad \zeta \gamma^2 < \zeta_2, \\ 0 < \zeta < \zeta_2 < \gamma \zeta_2 < \zeta_1 < \gamma \zeta_1 < \beta < \zeta_0 < r < \tau < 1 \quad \text{with } \zeta \gamma^2 < \zeta_2. \end{aligned} \quad (3.5)$$

τ will play the role of ς in Lemma 3.1, β will control our resonant boxes, and γ will control the growth of our length scales. We will let m^* denote the mass m_τ^* that we get from the induction hypothesis.

3.1. The first multiscale analysis.

Proposition 3.2. *Let $\theta > 8Nd$ and $E \in \mathbb{R}$. Take $0 < p < p + Nd < s < s + 2Nd - 2 < \theta$, $Y \geq 4000N^{N+1}$, and $p_0 = p_0(N) < \frac{1}{2}(2Y)^{-Nd}$. Then there exists a length scale Z_0^* such that if for some $L_0 \geq Z_0^*$ we have*

$$\sup_{\mathbf{x} \in \mathbb{R}^{Nd}} \mathbb{P}\left\{\Lambda_{L_0}^{(N)}(\mathbf{x}) \text{ is } (\theta, E)\text{-nonsuitable}\right\} \leq p_0, \quad (3.6)$$

then, setting $L_{k+1} = Y L_k$, for $k = 0, 1, 2, \dots$, there exists $K_0 \in \mathbb{N}$ such that for every $k \geq K_0$ we have

$$\sup_{\mathbf{x} \in \mathbb{R}^{Nd}} \mathbb{P}\left\{\Lambda_{L_k}^{(N)}(\mathbf{x}) \text{ is } (\theta, E)\text{-nonsuitable}\right\} \leq L_k^{-p}. \quad (3.7)$$

To prove the proposition we use the following deterministic lemma.

Lemma 3.3. *Let $\theta > 8Nd$ and $E \in \mathbb{R}$. Take $Nd < s < s + 2Nd - 2 < \theta$. Let $J \in \mathbb{N}$, $Y \geq 4000JN^{N+1}$, $L = Y\ell$, and $\mathbf{x} \in \mathbb{R}^{Nd}$. Suppose we have the following:*

- (i) $\Lambda_L^{(N)}(\mathbf{x})$ is E -suitably nonresonant.
- (ii) Every box $\Lambda_{K_j \ell}^{(N)}(\mathbf{u}) \subseteq \Lambda_L^{(N)}(\mathbf{x})$, with $\mathbf{u} \in \Xi_{L, \ell}(\mathbf{x})$, $j = 1, 2, \dots, JN^N$, where K_j is given in (2.29), is E -suitably nonresonant.
- (iii) There are at most J pairwise ℓ -distant, (E, θ) -nonsuitable boxes in the ℓ -suitable cover.

Then the N -particle box $\Lambda_L^{(N)}(\mathbf{x})$ is (E, θ) -suitable for L sufficiently large.

Proof. Since there are at most J pairwise ℓ -distant N -particle boxes in the suitable cover that are (E, θ) -nonsuitable, we can find $\mathbf{a}_s \in \Xi_{L, \ell}^{(N)}(\mathbf{x})$, $s = 1, 2, \dots, j \leq J$, such that the boxes $\Lambda_\ell(\mathbf{a}_s)$ are pairwise ℓ -distant, (E, θ) -nonsuitable boxes, and any box $\Lambda_\ell(\mathbf{a})$ with $\mathbf{a} \in \Xi_{L, \ell}^{(N)}(\mathbf{x})$ which is ℓ -distant from all the $\Lambda_\ell(\mathbf{a}_s)$ must be (E, θ) -suitable. Applying Lemma 2.15, we obtain $\tilde{\Upsilon} = \tilde{\Upsilon}_{L, \ell}^{(N)}(\{\mathbf{a}_s\}_{s=1}^J)$ as in (2.30),

satisfying the conclusions of that lemma. In particular, for $\mathbf{y} \in \Lambda_L^{(N)}(\mathbf{x}) \setminus \tilde{\Upsilon}$, the boxes $\Lambda_\ell^{(N, \mathbf{y})}$ and $\Lambda_\ell(\mathbf{a}_s)$ are ℓ -distant for $s = 1, 2, \dots, J$, and hence $\Lambda_\ell^{(N, \mathbf{y})}$ is a (E, θ) -suitable box.

Let $\mathbf{a} \in \Lambda_L^{(N)}(\mathbf{x})$. Then, either $\Lambda_\ell^{(\mathbf{a})}$ is (θ, E) -suitable or $\Lambda_\ell^{(\mathbf{a})}$ is (θ, E) -unsuitable. We proceed as follows:

- (i) If $\Lambda_\ell^{(\mathbf{a})}$ is (θ, E) -suitable, and $\mathbf{b} \in \Lambda_L^{(N)}(\mathbf{x}) \setminus \Lambda_\ell^{(\mathbf{a})}$, we use the resolvent identity to get

$$\begin{aligned} \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{a}, \mathbf{b}; E) \right| &\leq s_{Nd} \ell^{Nd-1-\theta} \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{v}', \mathbf{b}; E) \right| \text{ for some } \mathbf{v}' \in \partial_+ \Lambda_\ell^{(\mathbf{a})}, \\ \text{so } \frac{\ell}{10} &\leq \|\mathbf{v}' - \mathbf{a}\| \leq \ell + 1. \end{aligned} \quad (3.8)$$

- (ii) If $\Lambda_\ell^{(\mathbf{a})}$ is (θ, E) -unsuitable, we must have $\mathbf{a} \in \tilde{\Upsilon}$, and hence $\mathbf{a} \in \Lambda_{K_{j_t}\ell}^{(N)}(\mathbf{u}_t)$ for some t . Let $\mathbf{b} \in \Lambda_L^{(N)}(\mathbf{x}) \setminus \Lambda_{(K_{j_t}+2)\ell}^{(N)}(\mathbf{u}_t)$. Applying the resolvent identity, and using the fact that $\Lambda(t) := \Lambda_{K_{j_t}\ell}^{(N)}(\mathbf{u}_t)$ is E -suitably nonresonant by hypothesis, we get

$$\begin{aligned} \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{a}, \mathbf{b}; E) \right| &\leq |\partial \Lambda(t)| \left[\max_{(\mathbf{u}, \mathbf{v}) \in \partial \Lambda(t)} \left| G_{\Lambda(t)}(\mathbf{a}, \mathbf{u}; E) G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{v}, \mathbf{b}; E) \right| \right] \\ &\leq |\partial \Lambda(t)| (K_{j_t}\ell)^s \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{v}', \mathbf{b}; E) \right| \text{ for some } \mathbf{v}' \in \partial_+ \Lambda(t), \end{aligned} \quad (3.9)$$

where $\Lambda_\ell^{(\mathbf{v}')}$ is (θ, E) -suitable. We use (3.8) with $\mathbf{a} = \mathbf{v}'$, getting

$$\begin{aligned} \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{a}, \mathbf{b}; E) \right| &\leq s_{Nd}^2 L^{Nd-1+s} \ell^{Nd-1-\theta} \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{v}'', \mathbf{b}; E) \right| \\ &< \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{v}'', \mathbf{b}; E) \right| \text{ for some } \mathbf{v}'' \in \partial_+ \Lambda_\ell^{(\mathbf{v}')} \\ \text{so } \|\mathbf{v}'' - \mathbf{a}\| &\leq (\ell + 1) + (K_{j_t}\ell + 1) \leq 2K_{j_t}\ell, \end{aligned} \quad (3.10)$$

if we can guarantee $s_{Nd}^2 L^{Nd+s-1} \ell^{Nd-1-\theta} < 1$. Since $L = Y\ell$, we need

$$s_{Nd}^2 (Y\ell)^{Nd+s-1} \ell^{Nd-1-\theta} = s_{Nd}^2 Y^{Nd+s-1} \ell^{2Nd+s-2-\theta} \leq 1,$$

which is certainly true by our choice of s and θ provided that we take ℓ large enough.

Given $\mathbf{a}, \mathbf{b} \in \Lambda_L^{(N)}(\mathbf{x})$ with $\|\mathbf{a} - \mathbf{b}\| \geq \frac{L}{100}$, we estimate $\left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{a}, \mathbf{b}; E) \right|$ by, when possible, repeatedly using either (3.8) or (3.10), as appropriate, and, when we must stop because we got too close to \mathbf{b} , using the hypothesis that $\Lambda_L^{(N)}(\mathbf{x})$ is E -suitably nonresonant, obtaining

$$\left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{a}, \mathbf{b}; E) \right| \leq (s_{Nd} \ell^{Nd-1-\theta})^{N(Y)} L^s, \quad (3.11)$$

where $N(Y)$ is the number of times we used (3.8). We can always use either (3.8) or (3.10), unless we got too close to some \mathbf{v} where $\Lambda_\ell^{(\mathbf{v})}$ is (E, θ) -suitable and $\mathbf{b} \in \Lambda_\ell^{(\mathbf{v})}$, or $\mathbf{v} \in \Lambda_{K_{j_t}\ell}^{(N)}(\mathbf{u}_t)$ for some t and $\mathbf{b} \in \Lambda_{K_{j_t}\ell}^{(N)}(\mathbf{u}_t)$. It follows that we will not have to stop before

$$N(Y)(\ell + 1) + \sum_{t=1}^T 2K_{j_t}\ell + (\ell + 1) \geq \|\mathbf{b} - \mathbf{a}\| \ell \geq \frac{L}{100}. \quad (3.12)$$

Thus, using (2.33), we can achieve

$$N(Y) \geq \left(\frac{Y}{100} - 34JN^{N+1}\right) \frac{\ell}{\ell+1} - 2. \quad (3.13)$$

We take $Y \geq 4000JN^{N+1}$, which guarantees $N(Y) \geq 2$ for large ℓ by (3.13). It then follows from (3.11) that for $\mathbf{a}, \mathbf{b} \in \Lambda_L^{(N)}(\mathbf{x})$ with $\|\mathbf{a} - \mathbf{b}\| \geq \frac{L}{100}$ we have

$$\left|G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{a}, \mathbf{b}; E)\right| \leq L^{-\theta}, \quad (3.14)$$

and we conclude that $\Lambda_L^{(N)}(\mathbf{x})$ is (E, θ) -suitable for L sufficiently large. \square

Proof of Proposition 3.2. Given a scale L , we set

$$p_L = \sup_{\mathbf{x} \in \mathbb{R}^{Nd}} \mathbb{P}\left\{\Lambda_L^{(N)}(\mathbf{x}) \text{ is } (\theta, E)\text{-unsuitable}\right\}. \quad (3.15)$$

We assume $L = Y\ell$, with ℓ is sufficiently large when necessary.

Let $\Lambda_L^{(N)}(\mathbf{x})$ be an N -particle box with an ℓ -suitable cover, $\mathcal{C}_{L,\ell}(\mathbf{x})$, where $L = Y\ell$. Let $J \in 2\mathbb{N}$, to be specified later. We define several events: $\mathcal{E} = \left\{\Lambda_L^{(N)}(\mathbf{x}) \text{ is } (\theta, E)\text{-unsuitable}\right\}$, \mathcal{A} is the event that at least one of the PI boxes in $\mathcal{C}_{L,\ell}(\mathbf{x})$ is (θ, E) -unsuitable, \mathcal{W}_J is the event that (i) and (ii) in Lemma 3.3 hold, and \mathcal{F}_J is the event that (iii) in Lemma 3.3 holds. It follows from Lemma 3.3 that, taking $Y \geq 4000J^2N^{2N+1}$,

$$\mathbb{P}\{\mathcal{E}\} \leq \mathbb{P}\{\mathcal{W}_J^c\} + \mathbb{P}\{\mathcal{F}_J^c\} \leq \mathbb{P}\{\mathcal{W}_J^c\} + \mathbb{P}\{\mathcal{F}_J^c \cap \mathcal{A}^c\} + \mathbb{P}\{\mathcal{A}\}. \quad (3.16)$$

Lemma 3.1 yields

$$\mathbb{P}(\mathcal{A}) \leq (2Y)^{Nd} \ell^{Nd} e^{-\ell^\tau} \leq \frac{1}{4} L^{-p}. \quad (3.17)$$

Since $s > Nd + p$, Theorem 2.3 implies

$$\begin{aligned} \mathbb{P}(\mathcal{W}_J^c) &\leq 2N \|\rho\|_\infty \left(L^{Nd-s} + (2Y)^{Nd} \sum_{j=1}^{JN^N} (K_j \ell)^{Nd-s} \right) \\ &\leq 2N \|\rho\|_\infty (1 + (2Y)^s JN^N) L^{Nd-s} \\ &\leq 2N \|\rho\|_\infty (1 + (2Y)^s JN^N) L^{-p} \leq \frac{1}{4} L^{-p}. \end{aligned} \quad (3.18)$$

To estimate $\mathbb{P}\{\mathcal{F}_J^c \cap \mathcal{A}^c\}$, note that if $\omega \in \mathcal{F}_J^c \cap \mathcal{A}^c$, then there exist $J+1$ FI pairwise ℓ -distant boxes in the suitable cover that are (θ, E) -unsuitable. By Lemma 2.10 these boxes are fully separated. Thus

$$\mathbb{P}\{\mathcal{F}_J^c \cap \mathcal{A}^c\} \leq (2Y)^{(J+1)Nd} p_\ell^{J+1}. \quad (3.19)$$

Since $\mathbf{x} \in \mathbb{R}^{Nd}$ is arbitrary, we conclude that

$$p_L \leq \frac{1}{2} L^{-p} + (2Y)^{(J+1)Nd} p_\ell^{J+1}. \quad (3.20)$$

Since $J \geq 1$, it follows immediately from (3.20) that $p_\ell \leq \ell^{-p}$ implies $p_L \leq L^{-p}$.

We now fix L_0 , set $L_k = YL_{k-1}$ for $k = 1, 2, \dots$, and let $p_k = p_{L_k}$. To finish the proof, we need to show that, if $p_0 < \frac{1}{2} (2Y)^{-Nd}$, for sufficiently large L_0 , we have

$$K_0 = \inf \{k = 0, 1, \dots \mid p_k \leq L_k^{-p}\} < \infty. \quad (3.21)$$

It follows from equation (3.20) that

$$p_{k+1} \leq \frac{1}{2} L_{k+1}^{-p} + \left((2Y)^{Nd} p_k \right)^{J+1} \quad \text{for } k = 1, 2, \dots \quad (3.22)$$

If $k + 1 < K_0$, we conclude that

$$p_{k+1} < 2 \left((2Y)^{Nd} p_k \right)^{J+1} \quad (3.23)$$

If $K_0 = 0$ or $K_0 = 1$ we are done. If not, we have $p_1 \leq 2 \left((2Y)^{Nd} p_0 \right)^{J+1}$. If $K_0 > 2$, we have

$$p_2 \leq 2 \left((2Y)^{Nd} p_1 \right)^{J+1} \leq 2 \left((2Y)^{Nd} 2 \left((2Y)^{Nd} p_0 \right)^{J+1} \right)^{J+1} = \left(2 (2Y)^{Nd} \right)^{1+J} p_0^{(J+1)^2}. \quad (3.24)$$

Repeating this procedure, if $k < K_0$ we obtain

$$(Y^k L_0)^{-p} = L_k^{-p} < p_k \leq \left(2 (2Y)^{Nd} \right)^{\frac{(J+1)^k - 1}{(J+1) - 1}} p_0^{(J+1)^k}. \quad (3.25)$$

We now choose $J = 1$, obtaining

$$2 (2Y)^{Nd} (Y^k L_0)^{-p} < \left(2 (2Y)^{Nd} p_0 \right)^{2^k}. \quad (3.26)$$

We conclude that $K_0 < \infty$, since by hypothesis $2 (2Y)^{Nd} p_0 < 1$. \square

3.2. The second multiscale analysis.

Proposition 3.4. *Let $E \in \mathbb{R}$, $p > 0$, $0 < m_0 < m^*$, $1 < \gamma < 1 + \frac{p}{p+2Nd}$. Then there exists a length scale Z_1^* such that if for some $L_0 \geq Z_1^*$ we can verify*

$$\sup_{\mathbf{x} \in \mathbb{R}^{Nd}} \mathbb{P} \left\{ \Lambda_{L_0}^{(N)}(\mathbf{x}) \text{ is } (m_0, E)\text{-nonregular} \right\} \leq L_0^{-p}, \quad (3.27)$$

then, setting $L_{k+1} = L_k^\gamma$, for $k = 1, 2, \dots$, we get

$$\sup_{\mathbf{x} \in \mathbb{R}^{Nd}} \mathbb{P} \left\{ \Lambda_{L_k}^{(N)}(\mathbf{x}) \text{ is } \left(\frac{m_0}{2}, E \right)\text{-nonregular} \right\} \leq L_k^{-p} \quad \text{for all } k = 0, 1, 2, \dots \quad (3.28)$$

To prove the proposition we use the following deterministic lemma.

Lemma 3.5. *Let $E \in \mathbb{R}$, $L = \ell^\gamma$, $J \in \mathbb{N}$, $m_0 > 0$, and*

$$m_\ell \in \left[\frac{1}{\ell^\kappa}, m_0 \right], \quad \text{where } 0 < \kappa < \min \{ \gamma - 1, \gamma(1 - \beta), 1 \}. \quad (3.29)$$

Suppose that we have the following:

- (i) $\Lambda_L^{(N)}(\mathbf{x})$ is E -nonresonant.
- (ii) Every box $\Lambda_{K_j \ell}^{(N)}(\mathbf{u}) \subseteq \Lambda_L^{(N)}(\mathbf{x})$, with $\mathbf{u} \in \Xi_{L, \ell}(\mathbf{x})$, $j = 1, 2, \dots, JN^N$, where K_j is given in (2.29), is E -nonresonant.
- (iii) There are at most J pairwise ℓ -distant, (E, m_ℓ) -nonregular boxes in the suitable cover.

Then $\Lambda_L^{(N)}(\mathbf{x})$ is (E, m_L) -regular for L large, where

$$m_\ell \geq m_L \geq m_\ell - \frac{1}{2\ell^\kappa} \geq \frac{1}{L^\kappa}. \quad (3.30)$$

Proof. Since there are at most J pairwise ℓ -distant N -particle boxes in the suitable cover that are (E, m_ℓ) -nonregular, we can find $\mathbf{a}_s \in \Xi_{L, \ell}^{(N)}(\mathbf{x})$, $s = 1, 2, \dots, j \leq J$, such that the boxes $\Lambda_\ell(\mathbf{a}_s)$ are pairwise ℓ -distant, (E, m_ℓ) -nonregular boxes, and any box $\Lambda_\ell(\mathbf{a})$ with $\mathbf{a} \in \Xi_{L, \ell}^{(N)}(\mathbf{x})$ which is ℓ -distant from all the $\Lambda_\ell(\mathbf{a}_s)$ must be (E, m_ℓ) -regular. Proceeding as in the proof of Lemma 3.3, we obtain

$\tilde{\Upsilon} = \tilde{\Upsilon}_{L,\ell}^{(N)}(\{\mathbf{a}_s\}_{s=1}^J)$ as in (2.30), satisfying the conclusions of Lemma 2.15. In particular, for $\mathbf{y} \in \Lambda_L^{(N)}(\mathbf{x}) \setminus \tilde{\Upsilon}$, the boxes $\Lambda_\ell^{(N,\mathbf{y})}$ and $\Lambda_\ell(\mathbf{a}_s)$ are ℓ -distant for $s = 1, 2, \dots, J$, and hence $\Lambda_\ell^{(N,\mathbf{y})}$ is a (E, θ) -regular box.

Let $\mathbf{a} \in \Lambda_L^{(N)}(\mathbf{x})$. Then, either $\Lambda_\ell^{(\mathbf{a})}$ is (E, m_ℓ) -regular or $\Lambda_\ell^{(\mathbf{a})}$ is (E, m_ℓ) -nonregular. We proceed as follows:

- (i) If $\Lambda_\ell^{(\mathbf{a})}$ is (E, m_ℓ) -regular, and $\mathbf{b} \in \Lambda_L^{(N)}(\mathbf{x})$ with $\mathbf{b} \notin \Lambda_\ell^{(\mathbf{a})}$, then we use the resolvent identity as in (3.8) to get

$$\begin{aligned} \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{a}, \mathbf{b}; E) \right| &\leq \left| \partial \Lambda_\ell^{(\mathbf{a})} \right| \left[\max_{(\mathbf{u}, \mathbf{v}) \in \partial \Lambda_\ell^{(\mathbf{a})}} \left| G_{\Lambda_\ell^{(\mathbf{a})}}(\mathbf{a}, \mathbf{u}; E) G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{v}, \mathbf{b}; E) \right| \right] \\ &\leq s_{Nd} \ell^{Nd-1} e^{-m_\ell \|\mathbf{a} - \mathbf{b}'_1\|} \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{b}_1, \mathbf{b}; E) \right| \quad \text{for some } (\mathbf{b}'_1, \mathbf{b}_1) \in \partial \Lambda_\ell^{(\mathbf{a})} \\ &\leq s_{Nd} \ell^{Nd-1} e^{-m_\ell (\|\mathbf{a} - \mathbf{b}_1\| - 1)} \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{b}_1, \mathbf{b}; E) \right| \quad \text{for some } \mathbf{b}_1 \in \partial_+ \Lambda_\ell^{(\mathbf{a})} \\ &\leq e^{-m'_\ell \|\mathbf{a} - \mathbf{b}_1\|} \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{b}_1, \mathbf{b}; E) \right| \quad \text{for some } \mathbf{b}_1 \in \partial_+ \Lambda_\ell^{(\mathbf{a})}. \end{aligned} \quad (3.31)$$

Since $\frac{\ell}{10} \leq \|\mathbf{b}_1 - \mathbf{a}\| \leq \ell + 1$, and we assumed (3.29), this holds with

$$m'_\ell = (1 - \frac{10}{\ell})m_\ell - \frac{10}{\ell} \log(s_{Nd} \ell^{Nd-1}) \geq m_\ell - C_1(d, N, m_0) \frac{\log \ell}{\ell} > 0. \quad (3.32)$$

- (ii) If $\Lambda_\ell^{(\mathbf{a})}$ is (E, m_ℓ) -nonregular, we must have $\mathbf{a} \in \tilde{\Upsilon}$, and hence $\mathbf{a} \in \Lambda_{K_{j_t}\ell}^{(N)}(\mathbf{u}_t)$ for some t . Let $\mathbf{b} \in \Lambda_L^{(N)}(\mathbf{x}) \setminus \Lambda_{(K_{j_t}+2)\ell}^{(N)}(\mathbf{u}_t)$. Proceeding as in (3.9)-(3.10), and using (3.31), we get

$$\begin{aligned} \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{a}, \mathbf{b}; E) \right| &\leq s_{Nd} (K_{j_t}\ell)^{Nd-1} e^{(K_{j_t}\ell)^\beta} e^{-m'_\ell \frac{\ell}{10}} \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{v}'', \mathbf{b}; E) \right| \\ &< \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{v}'', \mathbf{b}; E) \right|, \end{aligned} \quad (3.33)$$

for some $\mathbf{v}'' \in \Lambda_L^{(N)}(\mathbf{x})$ with $\|\mathbf{v}'' - \mathbf{a}\| \leq (\ell + 1) + (K_{j_t}\ell + 1) \leq 2K_{j_t}\ell$, since we have

$$s_{Nd} (K_{j_t}\ell)^{Nd-1} e^{(K_{j_t}\ell)^\beta} e^{-m'_\ell \frac{\ell}{10}} \leq s_{Nd} (17JN^N\ell)^{Nd-1} e^{(17JN^N\ell)^\beta} e^{-m'_\ell \frac{\ell}{10}} < 1, \quad (3.34)$$

by our choice of m_ℓ , provided that we take ℓ large enough.

Given $\mathbf{a}, \mathbf{b} \in \Lambda_L^{(N)}(\mathbf{x})$ with $\|\mathbf{a} - \mathbf{b}\| \geq \frac{L}{100}$, we estimate $\left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{a}, \mathbf{b}; E) \right|$, using repeatedly (3.31) and (3.33), as appropriate, and when we must stop because we got too close to \mathbf{b} , using the hypothesis that $\Lambda_L^{(N)}(\mathbf{x})$ is E -NR. Similarly to [GK3, Proof of Lemma 3.11], we can find $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_R \in \Lambda_L^{(N)}(\mathbf{x})$, such that

$$\sum_{r=1}^{R-1} \|\mathbf{v}_r - \mathbf{v}_{r+1}\| + \sum_{t=1}^T 2K_{j_t}\ell + (\ell + 1) \geq \|\mathbf{b} - \mathbf{a}\|, \quad (3.35)$$

so

$$\sum_{r=1}^{R-1} \|\mathbf{v}_r - \mathbf{v}_{r+1}\| \geq \|\mathbf{b} - \mathbf{a}\| - 36JN^{N+1}\ell. \quad (3.36)$$

and we have

$$\begin{aligned} |G_{\Lambda_L(\mathbf{x})}(\mathbf{a}, \mathbf{b}; E)| &\leq \prod_{r=1}^{R-1} e^{-m'_\ell \|\mathbf{v}_r - \mathbf{v}_{r+1}\|} \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{v}_R, \mathbf{b}; E) \right| \\ &\leq e^{-m'_\ell \sum_{r=1}^{R-1} \|\mathbf{v}_r - \mathbf{v}_{r+1}\|} e^{L^\beta} \leq e^{-m'_\ell \|\mathbf{a} - \mathbf{b}\| + m'_\ell 36JN^{N+1}\ell + \ell^{\gamma\beta}} \leq e^{-m_L \|\mathbf{a} - \mathbf{b}\|}, \end{aligned} \quad (3.37)$$

where, using (3.29),

$$\begin{aligned} m_L &= m'_\ell - \frac{100}{L} (36JN^{N+1}m'_\ell\ell + \ell^{\gamma\beta}) \\ &\geq m_\ell - \frac{C_1(d, N, m_0) \log \ell}{\ell} - \frac{C_2(d, N, m_0, J)}{\ell^{\gamma-1}} - \frac{100}{\ell^{\gamma(1-\beta)}} \geq m_\ell - \frac{1}{2\ell^\kappa} \geq \frac{1}{2\ell^\kappa} \geq \frac{1}{L^\kappa}. \end{aligned} \quad (3.38)$$

We proved that $\Lambda_L^{(N)}(\mathbf{x})$ is (E, m_L) -regular for L large, with m_L as in (3.30). \square

Proof of Proposition 3.4. Given a scale L and $m_L > 0$, we set

$$p_L(m_L) = \sup_{\mathbf{x} \in \mathbb{R}^{Nd}} \mathbb{P} \left\{ \Lambda_L^{(N)}(\mathbf{x}) \text{ is } (m_L, E)\text{-nonregular} \right\}. \quad (3.39)$$

We start by showing that there exists \widetilde{Z}_1 such that if $p_\ell(m_\ell) \leq \ell^{-p}$, where m_ℓ satisfies (3.29), and $\ell \geq \widetilde{Z}_1$, then, setting $L = \ell^\gamma$, we have $p_L(m_L) \leq L^{-p}$ with m_L as in (3.30).

Let $\Lambda_L^{(N)}(\mathbf{x})$ be an N -particle box and $J \in \mathbb{N}$. We define several events: $\mathcal{E} = \left\{ \Lambda_L^{(N)}(\mathbf{x}) \text{ is } (m_L, E)\text{-nonregular} \right\}$, \mathcal{A} is the event that at least one of the PI boxes in $\mathcal{C}_{L, \ell}(\mathbf{x})$ is (m_L, E) -nonregular, \mathcal{W}_J is the event that (i) and (ii) in Lemma 3.5 hold, and \mathcal{F}_J is the event that (iii) in Lemma 3.5 holds. It follows from Lemma 3.5 that

$$\mathbb{P} \{ \mathcal{E} \} \leq \mathbb{P} \{ \mathcal{W}_J^c \} + \mathbb{P} \{ \mathcal{F}_J^c \} \leq \mathbb{P} \{ \mathcal{W}_J^c \} + \mathbb{P} \{ \mathcal{F}_J^c \cap \mathcal{A}^c \} + \mathbb{P} \{ \mathcal{A} \}. \quad (3.40)$$

Lemma 3.1 yields (large ℓ , so $m_\ell \leq m_\tau^*(\ell)$ in Lemma 3.1)

$$\mathbb{P}(\mathcal{A}) \leq \left(\frac{2L}{\ell} \right)^{Nd} \ell^{Nd} e^{-\ell^\tau} \leq \frac{1}{4} L^{-p}. \quad (3.41)$$

Theorem 2.3 implies

$$\mathbb{P}(\mathcal{W}_J^c) \leq N \|\rho\|_\infty \left(L^{Nd} e^{-L^\beta} + \left(\frac{2L}{\ell} \right)^{Nd} \sum_{j=1}^{JN^N} (K_j \ell)^{Nd} e^{-(K_j \ell)^\beta} \right) \quad (3.42)$$

$$\leq N \|\rho\|_\infty (1 + (2\ell^{\gamma-1})^{Nd} JN^N) e^{-\frac{1}{2}\ell^\beta} \leq \frac{1}{4} L^{-p}. \quad (3.43)$$

To estimate $\mathbb{P} \{ \mathcal{F}_J^c \cap \mathcal{A}^c \}$, note that if $\omega \in \mathcal{F}_J^c \cap \mathcal{A}^c$, then there exist $J+1$ FI pairwise ℓ -distant boxes in the suitable cover that are (θ, E) -nonregular. By Lemma 2.10 these boxes are fully separated. Thus

$$\mathbb{P} \{ \mathcal{F}_J^c \cap \mathcal{A}^c \} \leq \left(\frac{2L}{\ell} \right)^{(J+1)Nd} (p_\ell(m_\ell))^{J+1} \leq \frac{1}{2} L^{-p} + (2\ell^{\gamma-1})^{(J+1)Nd} \ell^{-(J+1)p}. \quad (3.44)$$

We now take $J = 1$, require $1 < \gamma < 1 + \frac{p}{p+2Nd}$ and conclude that, since $\mathbf{x} \in \mathbb{R}^{Nd}$ is arbitrary,

$$p_L(m_L) \leq \frac{1}{2} L^{-p} + (2\ell^{\gamma-1})^{2Nd} \ell^{-2p} \leq \frac{1}{2} L^{-p} + \frac{1}{2} L^{-p} \leq L^{-p}. \quad (3.45)$$

We now fix L_0 and $m_0 > 0$. We take L_0 is sufficiently large, so $m_0 \geq L_0^{-\kappa}$. We set $L_k = L_{k-1}^\gamma$ and $m_k = m_{k-1} - \frac{1}{2L_{k-1}^\kappa}$ for $k = 1, 2, \dots$, and let $p_k = p_{L_k}(m_k)$. If

$p_0 \leq L_0^{-p}$, we conclude that $p_k \leq L_k^{-p}$ for $k = 1, 2, \dots$. Moreover,

$$m_0 - m_k \leq \sum_{j=1}^{\infty} (m_{j-1} - m_j) = \frac{1}{2} \sum_{j=1}^{\infty} L_{j-1}^{-\kappa} = \frac{1}{2} \sum_{j=1}^{\infty} L_0^{-\kappa \gamma^{j-1}} \leq \frac{m_0}{2}, \quad (3.46)$$

so $m_k \geq \frac{m_0}{2}$ for $k = 1, 2, \dots$ \square

3.3. The third multiscale analysis.

Proposition 3.6. *Let $0 < \zeta_1 < \zeta_0 < 1$ as in (3.5), $E \in \mathbb{R}$, and assume $Y \geq (3800N^{N+1})^{\frac{1}{1-\zeta_0}}$. Then there exists $Z_2^* > L_\tau$ such that, if for some scale $L_0 > Z_2^*$ we have*

$$\sup_{\mathbf{x} \in \mathbb{R}^{Nd}} \mathbb{P} \left\{ \mathbf{\Lambda}_{L_0}^{(N)}(\mathbf{x}) \text{ is } (\zeta_0, E)\text{-nonSES} \right\} \leq \left(2(2Y)^{Nd} \right)^{-\frac{1}{Y^{\zeta_0}-1}}, \quad (3.47)$$

then, setting $L_{k+1} = Y L_k$, $k = 0, 1, 2, \dots$, there exists $K_1 \in \mathbb{N}$ such that for every $k \geq K_1$ we have

$$\sup_{\mathbf{x} \in \mathbb{R}^{Nd}} \mathbb{P} \left\{ \mathbf{\Lambda}_{L_k}(\mathbf{x}) \text{ is } (\zeta_0, E)\text{-nonSES} \right\} \leq e^{-L_k^{\zeta_1}}. \quad (3.48)$$

As a consequence, for every $k \geq K_1$, we have

$$\sup_{\mathbf{x} \in \mathbb{R}^{Nd}} \mathbb{P} \left\{ \mathbf{\Lambda}_{L_k}(\mathbf{x}) \text{ is } (L_k^{\zeta_0-1}, E)\text{-nonregular} \right\} \leq e^{-L_k^{\zeta_1}}. \quad (3.49)$$

To prove the proposition, we use the following deterministic lemma.

Lemma 3.7. *Let $L = Y\ell$, where $Y \geq (3800N^{N+1})^{\frac{1}{1-\zeta_0}}$, and set $J = \lfloor Y^{\zeta_0} \rfloor$, the largest integer $\leq Y^{\zeta_0}$. Suppose the following are true:*

- (i) $\mathbf{\Lambda}_L^{(N)}(\mathbf{x})$ is E -nonresonant.
- (ii) Every box $\mathbf{\Lambda}_{K_j \ell}^{(N)}(\mathbf{u}) \subseteq \mathbf{\Lambda}_L^{(N)}(\mathbf{x})$, with $\mathbf{u} \in \Xi_{L, \ell}(\mathbf{x})$, $j = 1, 2, \dots, JN^N$, where K_j is given in (2.29), is E -nonresonant.
- (iii) There are at most J pairwise ℓ -distant, (E, ζ_0) -nonSES boxes in the suitable cover.

Then $\mathbf{\Lambda}_L^{(N)}(\mathbf{x})$ is (E, ζ_0) -SES, provided ℓ is sufficiently large.

Proof. Since there at most J pairwise ℓ -distant N -particle boxes in the suitable cover that are (E, ζ_0) -nonSES, we can find $\mathbf{a}_s \in \Xi_{L, \ell}^{(N)}(\mathbf{x})$, $s = 1, 2, \dots, j \leq J$, such that the boxes $\mathbf{\Lambda}_\ell(\mathbf{a}_s)$ are pairwise ℓ -distant, (E, ζ_0) -nonSES boxes, and any box $\mathbf{\Lambda}_\ell(\mathbf{a})$ with $\mathbf{a} \in \Xi_{L, \ell}^{(N)}(\mathbf{x})$ which is ℓ -distant from all the $\mathbf{\Lambda}_\ell(\mathbf{a}_s)$ must be (E, ζ_0) -SES.

Applying Lemma 2.15, we obtain $\tilde{\Upsilon} = \tilde{\Upsilon}_{L, \ell}^{(N)}(\{\mathbf{a}_s\}_{s=1}^J)$ as in (2.30), satisfying the conclusions of that lemma. In particular, for $\mathbf{y} \in \mathbf{\Lambda}_L^{(N)}(\mathbf{x}) \setminus \tilde{\Upsilon}$, the boxes $\mathbf{\Lambda}_\ell^{(N, \mathbf{y})}$ and $\mathbf{\Lambda}_\ell(\mathbf{a}_s)$ are ℓ -distant for $s = 1, 2, \dots, J$, and hence $\mathbf{\Lambda}_\ell^{(N, \mathbf{y})}$ is a (E, ζ_0) -SES box.

Let $\mathbf{a} \in \mathbf{\Lambda}_L^{(N)}(\mathbf{x})$. Then, either $\mathbf{\Lambda}_\ell^{(\mathbf{a})}$ is (E, ζ_0) -SES or $\mathbf{\Lambda}_\ell^{(\mathbf{a})}$ is (E, ζ_0) -nonSES. We proceed as follows:

- (i) If $\mathbf{\Lambda}_\ell^{(\mathbf{a})}$ is (E, ζ_0) -SES, and $\mathbf{b} \in \mathbf{\Lambda}_L^{(N)}(\mathbf{x}) \setminus \mathbf{\Lambda}_\ell^{(\mathbf{a})}$, we use the resolvent identity to get

$$\begin{aligned} \left| G_{\mathbf{\Lambda}_L^{(N)}(\mathbf{x})}(\mathbf{a}, \mathbf{b}; E) \right| &\leq s_{Nd} \ell^{Nd-1} e^{-\ell^{\zeta_0}} \left| G_{\mathbf{\Lambda}_L^{(N)}(\mathbf{x})}(\mathbf{v}', \mathbf{b}; E) \right| \text{ for some } \mathbf{v}' \in \partial_+ \mathbf{\Lambda}_\ell^{(\mathbf{a})}, \\ \text{so } \frac{\ell}{10} &\leq \|\mathbf{v}' - \mathbf{a}\| \leq \ell + 1. \end{aligned} \quad (3.50)$$

(ii) If $\Lambda_\ell^{(\mathbf{a})}$ is (E, ζ_0) -nonSES, we must have $\mathbf{a} \in \tilde{\Upsilon}$, and hence $\mathbf{a} \in \Lambda_{K_{j_t}\ell}^{(N)}(\mathbf{u}_t)$ for some t . Let $\mathbf{b} \in \Lambda_L^{(N)}(\mathbf{x}) \setminus \Lambda_{(K_{j_t}+2)\ell}^{(N)}(\mathbf{u}_t)$. Applying the resolvent identity, and using the fact that $\Lambda(t) := \Lambda_{K_{j_t}\ell}^{(N)}(\mathbf{u}_t)$ is E -nonresonant by hypothesis, we get

$$\begin{aligned} \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{a}, \mathbf{b}; E) \right| &\leq |\partial\Lambda(t)| \left[\max_{(\mathbf{u}, \mathbf{v}) \in \partial\Lambda(t)} \left| G_{\Lambda(t)}(\mathbf{a}, \mathbf{u}; E) G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{v}, \mathbf{b}; E) \right| \right] \\ &\leq s_{Nd} (K_{j_t}\ell)^{Nd-1} e^{(K_{j_t}\ell)^\beta} \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{v}', \mathbf{b}; E) \right| \text{ for some } \mathbf{v}' \in \partial_+\Lambda(t), \end{aligned} \quad (3.51)$$

where $\Lambda_\ell^{(\mathbf{v}')}$ is (E, ζ_0) -SES. We use (3.50) with $\mathbf{a} = \mathbf{v}'$, getting

$$\begin{aligned} \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{a}, \mathbf{b}; E) \right| &\leq s_{Nd}^2 L^{2(Nd-1)} e^{(17JN^{N+1}\ell)^\beta} e^{-\ell\zeta_0} \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{v}'', \mathbf{b}; E) \right| \\ &\leq \left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{v}'', \mathbf{b}; E) \right| \text{ for some } \mathbf{v}'' \in \partial_+\Lambda_\ell^{(\mathbf{v}')}, \\ \text{so } \|\mathbf{v}'' - \mathbf{a}\| &\leq (\ell + 1) + (K_{j_t}\ell + 1) < 18JN^{N+1}\ell, \end{aligned} \quad (3.52)$$

if we can guarantee $s_{Nd}^2 L^{2(Nd-1)} e^{(17JN^{N+1}\ell)^\beta} e^{-\ell\zeta_0} < 1$. Since $L = Y\ell$, we need

$$s_{Nd}^2 Y^{2(Nd-1)} \ell^{2(Nd-1)} e^{(17JN^{N+1}\ell)^\beta} e^{-\ell\zeta_0} \leq 1,$$

which is certainly true since $\beta < \zeta_0$, provided that we take ℓ large enough.

Given $\mathbf{a}, \mathbf{b} \in \Lambda_L^{(N)}(\mathbf{x})$ with $\|\mathbf{a} - \mathbf{b}\| \geq \frac{L}{100}$, we estimate $\left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{a}, \mathbf{b}; E) \right|$ by, when possible, repeatedly using either (3.50) or (3.52), as appropriate, and, when we must stop because we got too close to \mathbf{b} , using the hypothesis that $\Lambda_L^{(N)}(\mathbf{x})$ is E -NR, obtaining

$$\left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{a}, \mathbf{b}; E) \right| \leq \left(s_{Nd} \ell^{Nd-1} e^{-\ell\zeta_0} \right)^{N(Y)} e^{L^\beta}, \quad (3.53)$$

where $N(Y)$ is the number of times we used (3.50). We can always use either (3.50) or (3.52), unless we got to some \mathbf{v} where $\Lambda_\ell^{(\mathbf{v})}$ is (E, ζ_0) -SES and $\mathbf{b} \in \Lambda_\ell^{(\mathbf{v})}$, or $\mathbf{v} \in \Lambda_{K_{j_t}\ell}^{(N)}(\mathbf{u}_t)$ for some t and $\mathbf{b} \in \Lambda_{K_{j_t}\ell}^{(N)}(\mathbf{u}_t)$. As in the proof of Lemma 3.3, we have (3.12) and (3.13).

If we have

$$N(Y) \geq 2Y^{\zeta_0}, \quad (3.54)$$

it follows from (3.53) that for $\mathbf{a}, \mathbf{b} \in \Lambda_L^{(N)}(\mathbf{x})$ with $\|\mathbf{a} - \mathbf{b}\| \geq \frac{L}{100}$, and L sufficiently large, we have

$$\left| G_{\Lambda_L^{(N)}(\mathbf{x})}(\mathbf{a}, \mathbf{b}; E) \right| \leq e^{-L^{\zeta_0}}, \quad (3.55)$$

and we conclude that $\Lambda_L^{(N)}(\mathbf{x})$ is (E, ζ_0) -SES.

To finish the proof we need to show that we can guarantee (3.54) for large ℓ . It follows from (3.13) that it suffices to have $Y \geq 200(18JN^{N+1} + Y^{\zeta_0})$. We fix $J = \lfloor Y^{\zeta_0} \rfloor$, the largest integer $\leq Y^{\zeta_0}$, so it suffices to require $Y \geq (3800N^{N+1})^{\frac{1}{1-\zeta_0}}$ to get (3.54). \square

Proof of Proposition 3.6. Given a scale L , we set

$$p_L = \sup_{\mathbf{x} \in \mathbb{R}^{Nd}} \mathbb{P} \left\{ \Lambda_L^{(N)}(\mathbf{x}) \text{ is } (\zeta_0, E)\text{-nonSES} \right\}. \quad (3.56)$$

We assume $L = Y\ell$, with ℓ is sufficiently large when necessary.

We proceed as in the proof of Proposition 3.2. Let $\Lambda_L^{(N)}(\mathbf{x})$ be an N -particle box with an ℓ -suitable cover, $\mathcal{C}_{L,\ell}(\mathbf{x})$, where $L = Y\ell$. Assume $Y \geq (3800N^{N+1})^{\frac{1}{1-\zeta_0}}$, and set $J = \lfloor Y^{\zeta_0} \rfloor$. We define events $\mathcal{E}, \mathcal{A}, \mathcal{W}_J, \mathcal{F}_J$ as in the proof of Proposition 3.2, with (ζ_0, E) -nonSES boxes instead of (θ, E) -unsuitable boxes, etc. It follows from Lemma 3.7 that (3.16) holds. Using Lemma 3.1 we get

$$\mathbb{P}(\mathcal{A}) \leq (2Y)^{Nd} \ell^{Nd} e^{-\ell^\tau} \leq \frac{1}{4} e^{-L^{\zeta_1}}. \quad (3.57)$$

Proceeding as in (3.42), with our choice of β Theorem 2.3 implies

$$\mathbb{P}(\mathcal{W}_J^c) \leq N \|\rho\|_\infty (1 + (2\ell^{\gamma-1})^{Nd} J N^N) e^{-\frac{1}{2}\ell^\beta} \leq \frac{1}{4} e^{-L^{\zeta_1}}. \quad (3.58)$$

We also have (3.19), so, similarly to (3.20), we get

$$p_L \leq \frac{1}{2} e^{-L^{\zeta_1}} + (2Y)^{(J+1)Nd} p_\ell^{J+1}. \quad (3.59)$$

Since $J+1 = \lfloor Y^{\zeta_0} \rfloor + 1 > Y^{\zeta_0} > Y^{\zeta_1}$, it follows immediately from (3.59) that $p_\ell \leq e^{-\ell^{\zeta_1}}$ implies $p_L \leq e^{-L^{\zeta_1}}$.

We now fix L_0 , set $L_k = YL_{k-1}$ for $k = 1, 2, \dots$, and let $p_k = p_{L_k}$. To finish the proof, we need to show that, if $p_0 < \frac{1}{2} (2Y)^{-Nd}$, for sufficiently large L_0 , we have

$$K_0 = \inf \left\{ k = 0, 1, \dots \mid p_k \leq e^{-L^{\zeta_1}} \right\} < \infty. \quad (3.60)$$

It follows from equation (3.59) that

$$p_{k+1} \leq \frac{1}{2} e^{-L_{k+1}^{\zeta_1}} + \left((2Y)^{Nd} p_k \right)^{J+1} \quad \text{for } k = 1, 2, \dots \quad (3.61)$$

If $k+1 < K_0$, we conclude that

$$p_{k+1} < 2 \left((2Y)^{Nd} p_k \right)^{J+1}. \quad (3.62)$$

If $K_0 = 0$ or $K_0 = 1$ we are done. If not, we have $p_1 \leq 2 \left((2Y)^{Nd} p_0 \right)^{J+1}$. If $K_0 > 2$ and $k < K_0$, proceeding as in (3.24)-(3.62) we get

$$e^{-(Y^k L_0)^{\zeta_1}} = e^{-L_k^{\zeta_1}} < p_k \leq \left(2 (2Y)^{Nd} \right)^{\frac{(J+1)^k - 1}{(J+1) - 1}} p_0^{(J+1)^k}. \quad (3.63)$$

Since $Y^{\zeta_0} - 1 < J = \lfloor Y^{\zeta_0} \rfloor \leq Y^{\zeta_0}$, and we assume $\left(2 (2Y)^{Nd} \right)^{\frac{1}{Y^{\zeta_0} - 1}} p_0 < 1$, we get

$$e^{-Y^{k\zeta_1} L_0^{\zeta_1}} \leq \left(\left(2 (2Y)^{Nd} \right)^{\frac{1}{Y^{\zeta_0} - 1}} p_0 \right)^{Y^{k\zeta_0}}. \quad (3.64)$$

Since $\zeta_0 > \zeta_1$ and $\left(2 (2Y)^{Nd} \right)^{\frac{1}{Y^{\zeta_0} - 1}} p_0 < 1$, we conclude that $K_0 < \infty$. \square

3.4. The fourth multiscale analysis. We fix $\zeta, \tau, \beta, \zeta_1, \zeta_2, \gamma$ as in (3.5).

3.4.1. The single energy multiscale analysis.

Proposition 3.8. *Let $0 < m_0 < m^* = m_\tau$. Then there exists a length scale Z_3^* such that, given an energy $E \in \mathbb{R}$, if for some $L_0 \geq Z_3^*$ we can verify*

$$\sup_{\mathbf{a} \in \mathbb{R}^{Nd}} \mathbb{P} \left\{ \mathbf{\Lambda}_{L_0}^{(N)}(\mathbf{a}) \text{ is } (m_0, E) \text{-nonregular} \right\} \leq e^{-L_0^{\zeta_2}}, \quad (3.65)$$

then for sufficiently large L we have

$$\sup_{\mathbf{a} \in \mathbb{R}^{Nd}} \mathbb{P} \left\{ \mathbf{\Lambda}_L^{(N)}(\mathbf{a}) \text{ is } \left(\frac{m_0}{2}, E\right) \text{-nonregular} \right\} \leq e^{-L^{\zeta_2}}. \quad (3.66)$$

Proposition 3.8 is proved first for a sequence of length scale L_k similarly to Proposition 3.4; to obtain the sub-exponential decay of probabilities we choose J , the number of bad boxes, dependent on the scale L as in the proof of Proposition 3.19 below. To obtain Proposition 3.8 as stated, that is, for all sufficiently large scales, we prove a slightly more general result.

Definition 3.9. *Let $E \in \mathbb{R}$. An N -particle box, $\mathbf{\Lambda}_L^{(N)}(\mathbf{x})$, is said to be (E, m_L) -good if and only if it is (E, m_L) -regular and E -nonresonant.*

Lemma 3.10. *Let $\mathbf{\Lambda}_L^{(N)}(\mathbf{x})$ be an N -particle box, $\gamma > 1$, $\ell = L^{\frac{1}{\gamma}}$ with $\gamma \leq \gamma' \leq \gamma^2$, and $m > 0$. Suppose every box in $\mathcal{C}_{L, \ell}^{(N)}(\mathbf{x})$ is (E, m) -good. Then $\mathbf{\Lambda}_L^{(N)}(\mathbf{x})$ is $(E, \frac{m}{2})$ -good.*

This lemma is a straightforward adaptation of [GK3, Lemma 3.16] to the discrete case.

Lemma 3.11. *Let $E_1 \in \mathbb{R}$, $\zeta_2 \in (\zeta, \tau)$, and $\gamma \in (1, \frac{1}{\zeta_2})$ with $\zeta \gamma^2 < \zeta_2$. Assume there exists a mass $m_{\zeta_2} > 0$ and a length scale $L_0 = L_0(\zeta_2)$, such that, taking $L_{k+1} = L_k^\gamma$ for $k = 0, 1, \dots$, we have*

$$\sup_{\mathbf{a} \in \mathbb{R}^{Nd}} P \left\{ \mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{a}) \text{ is not } (m_{\zeta_2}, E_1) \text{-good} \right\} \leq e^{-L_k^{\zeta_2}} \quad \text{for } k = 0, 1, \dots \quad (3.67)$$

Then there exists L_ζ such that for every $L \geq L_\zeta$ we have

$$\sup_{\mathbf{a} \in \mathbb{R}^{Nd}} P \left\{ \mathbf{\Lambda}_L^{(N)}(\mathbf{a}) \text{ is not } (m_{\zeta_2}, E_1) \text{-good} \right\} \leq e^{-L^\zeta}. \quad (3.68)$$

Proof. Given a scale L we take K such that $L_K \leq L < L_{K+1}$, and set $\ell = L_{K-1}$. Note that $L_K = \ell^\gamma$ and $L_{K+1} = L_K^\gamma = \ell^{\gamma^2}$, so $L = \ell^{\gamma'}$ with $\gamma \leq \gamma' < \gamma^2$.

Given an N -particle box $\mathbf{\Lambda}_L^{(N)}(\mathbf{x})$, let

$$\mathcal{F}_1 = \bigcup_{\mathbf{u} \in \Xi_{L, \ell}(\mathbf{x})} \mathcal{R}_{\mathbf{u}}, \quad \text{where } \mathcal{R}_{\mathbf{u}} = \left\{ \mathbf{\Lambda}_\ell^{(N)}(\mathbf{u}) \text{ is not } (m_{\zeta_2}, E_1) \text{-good} \right\}. \quad (3.69)$$

If $\omega \notin \mathcal{F}_1$, every box in $\mathcal{C}_{L, \ell}^{(N)}(\mathbf{x})$ is (m_{ζ_2}, E_1) -good, and hence $\mathbf{\Lambda}_L^{(N)}(\mathbf{x})$ is $(\frac{m_{\zeta_2}}{2}, E_1)$ -good by Lemma 3.10. The lemma follows since

$$\mathbb{P}(\mathcal{F}_1) \leq \left(\frac{2L}{\ell}\right)^{Nd} e^{-\ell^{\zeta_2}} \leq e^{-L^\zeta}. \quad (3.70)$$

□

3.4.2. The energy interval multiscale analysis.

Lemma 3.12. *Let $\Lambda_L^{(N)}(\mathbf{x})$ be an N -particle box and $m > 0$. Let $E_0 \in \mathbb{R}$, and suppose that*

- (i) $\Lambda_L^{(N)}(\mathbf{x})$ is (m, E_0) -regular,
- (ii) $\text{dist}\left(\sigma\left(H_{\Lambda_L^{(N)}(\mathbf{x})}\right), E_0\right) \geq e^{-L^\beta}$, i.e., $\left\|G_{\Lambda_L^{(N)}(\mathbf{x})}(E_0)\right\| \leq e^{L^\beta}$.

Then $\Lambda_L^{(N)}(\mathbf{x})$ is $\left(m - \frac{100 \log 2}{L}, E\right)$ -good for every $E \in I = (E_0 - \eta, E_0 + \eta)$, where

$$\eta = \frac{1}{2}e^{-mL-2L^\beta}. \quad (3.71)$$

Proof. Let $|E - E_0| \leq \frac{1}{2}e^{-L^\beta}$, so assumption (ii) implies

$$\text{dist}\left(\sigma\left(H_{\Lambda_L^{(N)}(\mathbf{x})}\right), E\right) \geq \frac{1}{2}e^{-L^\beta}, \quad \text{i.e.,} \quad \left\|G_{\Lambda_L^{(N)}(\mathbf{x})}(E)\right\| \leq 2e^{L^\beta}. \quad (3.72)$$

The resolvent equation gives

$$G_{\Lambda_L^{(N)}(\mathbf{x})}(E) = G_{\Lambda_L^{(N)}(\mathbf{x})}(E_0) + (E - E_0) G_{\Lambda_L^{(N)}(\mathbf{x})}(E) G_{\Lambda_L^{(N)}(\mathbf{x})}(E_0), \quad (3.73)$$

so for all $\mathbf{a}, \mathbf{b} \in \Lambda_L^{(N)}(\mathbf{x})$ we have

$$\begin{aligned} \left|G_{\Lambda_L^{(N)}(\mathbf{x})}(E; \mathbf{a}, \mathbf{b})\right| &\leq e^{-m\|\mathbf{a}-\mathbf{b}\|} + |E - E_0| \left\|G_{\Lambda_L^{(N)}(\mathbf{x})}(E)\right\| \left\|G_{\Lambda_L^{(N)}(\mathbf{x})}(E_0)\right\| \\ &\leq e^{-m\|\mathbf{a}-\mathbf{b}\|} + 2e^{2L^\beta} |E - E_0|. \end{aligned} \quad (3.74)$$

Now let $E \in I = (E_0 - \eta, E_0 + \eta)$, where η is as in (3.71). Since

$$\eta < \frac{1}{2}e^{-L^\beta} \quad \text{and} \quad \eta 2e^{2L^\beta} = e^{-mL} \leq e^{-m\|\mathbf{a}-\mathbf{b}\|} \quad (3.75)$$

we conclude that if $\|\mathbf{a} - \mathbf{b}\| \geq \frac{L}{100}$ we have

$$\left|G_{\Lambda_L^{(N)}(\mathbf{x})}(E; \mathbf{a}, \mathbf{b})\right| \leq 2e^{-m\|\mathbf{a}-\mathbf{b}\|} \leq e^{-(m - \frac{100 \log 2}{L})\|\mathbf{a}-\mathbf{b}\|}. \quad (3.76)$$

□

Proposition 3.6, combined with Theorem 2.3 and Lemma 3.12, yields the following proposition.

Proposition 3.13. *Let $0 < \zeta_2 < \zeta_1 < \zeta_0 < 1$, and assume the conclusions of Proposition 3.6. There exists scales L_k , $k = 1, 2, \dots$, such that $\lim_{k \rightarrow \infty} L_k = \infty$, with the following property: Let*

$$m_k = \left(L_k^{\zeta_0-1} - \frac{100 \log 2}{L_k}\right) \quad \text{and} \quad \eta_k = \frac{1}{2}e^{-L_k^{\zeta_0}-2L_k^\beta}. \quad (3.77)$$

Then for all $E_0 \in \mathbb{R}$ we have

$$\sup_{\mathbf{x} \in \mathbb{R}^{Nd}} \mathbb{P}\{\exists E \in (E_0 - \eta_k, E_0 + \eta_k) \text{ such that } \Lambda_{L_k}(\mathbf{x}) \text{ is } (m_k, E) \text{-nonregular}\} \leq e^{-L_k^{\zeta_1}}, \quad (3.78)$$

and

$$\begin{aligned} \sup_{\mathbf{x} \in \mathbb{R}^{Nd}} \mathbb{P}\{\exists E \in (E_0 - \eta_k, E_0 + \eta_k) \text{ such that } \Lambda_{L_k}(\mathbf{x}) \text{ is not } (m_k, E) \text{-good}\} \\ \leq e^{-L_k^{\zeta_1}} + 2N \|\rho\|_\infty L_k^{Nd} e^{-L_k^\beta} \leq e^{-L_k^{\zeta_2}}. \end{aligned} \quad (3.79)$$

We now take $L = \ell^\gamma$.

Definition 3.14. Let $\Lambda_L^{(N)}(\mathbf{x}) = \Lambda_L(\mathbf{x}_{\mathcal{J}}) \times \Lambda_L(\mathbf{x}_{\mathcal{J}^c})$ be a PI N -particle box with the usual ℓ suitable cover, and consider an energy $E \in \mathbb{R}$. Then:

- (i) $\Lambda_L^{(N)}(\mathbf{x})$ is not E -Lregular (for left regular) if and only if there are two partially separable boxes in $\mathcal{C}_{L,\ell}^{\mathcal{J}}(\mathbf{x}_{\mathcal{J}})$ that are $(m^*, E - \mu)$ -nonregular for some $\mu \in \sigma(H_{\Lambda_L^{\mathcal{J}^c}(\mathbf{x}_{\mathcal{J}^c})})$.
- (ii) $\Lambda_L^{(N)}(\mathbf{x})$ is not E -Rregular (for right regular) if and only if there are two partially separable boxes in $\mathcal{C}_{L,\ell}^{\mathcal{J}^c}(\mathbf{x}_{\mathcal{J}^c})$ that are $(m^*, E - \lambda)$ -nonregular for some $\lambda \in \sigma(H_{\Lambda_L(\mathbf{x}_{\mathcal{J}})})$.
- (iii) $\Lambda_L^{(N)}(\mathbf{x})$ is E -preregular if and only if $\Lambda_L^{(N)}(\mathbf{x})$ is E -Lregular and E -Rregular.

Lemma 3.15. Let $E_0 \in \mathbb{R}$, $I = [E_0 - \delta_\tau, E_0 + \delta_\tau]$, and consider a PI N -particle box $\Lambda_L^{(N)}(\mathbf{x}) = \Lambda_L(\mathbf{x}_{\mathcal{J}}) \times \Lambda_L(\mathbf{x}_{\mathcal{J}^c})$. Then

- (i) $\mathbb{P}\left\{\Lambda_L^{(N)}(\mathbf{x}) \text{ is not } E\text{-Lregular for some } E \in I\right\} \leq L^{3Nd}e^{-\ell^\tau},$
- (ii) $\mathbb{P}\left\{\Lambda_L^{(N)}(\mathbf{x}) \text{ is not } E\text{-Lregular for some } E \in I\right\} \leq L^{3Nd}e^{-\ell^\tau}.$

We conclude that

$$\mathbb{P}\{\Lambda_L^{(N)}(\mathbf{x}) \text{ is not } E\text{-preregular for some } E \in I\} \leq 2L^{3Nd}e^{-\ell^\tau}. \quad (3.80)$$

Proof. We prove (i), the proof of (ii) is similar. Let us set $\mathcal{S} = \Pi_{\mathcal{J}}\Lambda_L^{(N)}(\mathbf{x})$ and $\mathcal{B} = \left\{\exists E \in I \text{ such that } \Lambda_L^{(N)}(\mathbf{x}) \text{ is not } E\text{-Lregular}\right\}$. Since $\Lambda_L^{(N)}(\mathbf{x})$ is PI, we have $\mathbb{P}(\mathcal{B}) = \mathbb{E}_{\mathcal{S}^c}\mathbb{P}_{\mathcal{S}}(\mathcal{B})$.

Let us fix $\omega_{\mathcal{S}^c}$, and pick $\mu \in \sigma(H_{\omega_{\mathcal{S}^c}, \Lambda_L^{\mathcal{J}^c}(\mathbf{x}_{\mathcal{J}^c})})$. Let \mathcal{D} denote the event that there exists $E \in I$ such that $\Lambda_L(\mathbf{x}_{\mathcal{J}})$ contains two partially separable boxes in the ℓ -suitable cover that are $(E - \mu, m^*)$ -nonregular. We can rewrite \mathcal{D} as the event that there exists $E' \in I - \mu$ such that $\Lambda_L(\mathbf{x}_{\mathcal{J}})$ contains a pair of partially separable boxes in the ℓ -suitable cover that is (E', m^*) -nonregular, where $I - \mu = \left\{\mathcal{E} - \mu \mid \mathcal{E} \in I\right\}$. Applying the bootstrap MSA result to the interval $I - \mu$ for $|\mathcal{J}|$ particles (induction hypothesis), we get $\mathbb{P}(\mathcal{D}) \leq \left(\frac{2L}{\ell}\right)^{2|\mathcal{J}|d} e^{-\ell^\tau}$. We conclude that

$$\mathbb{P}(\mathcal{B}) \leq |\Lambda_L(\mathbf{a}_{\mathcal{J}^c})| \left(\frac{2L}{\ell}\right)^{2|\mathcal{J}|d} e^{-\ell^\tau} \leq L^{3Nd}e^{-\ell^\tau}. \quad (3.81)$$

□

Definition 3.16. Let $\Lambda_L^{(N)}(\mathbf{u}) = \Lambda_L(\mathbf{u}_{\mathcal{J}}) \times \Lambda_L(\mathbf{u}_{\mathcal{J}^c})$ be a PI N -particle box, and consider an energy $E \in \mathbb{R}$. Then:

- (i) $\Lambda_L^{(N)}(\mathbf{u})$ is E -left nonresonant (or LNR) if and only if for every box $\Lambda_{K_j\ell}(\mathbf{a}) \subseteq \Lambda_L(\mathbf{u}_{\mathcal{J}})$, with $\mathbf{a} \in \Xi_{L,\ell}(\mathbf{u}_{\mathcal{J}})$ and $j \in \left\{1, 2, \dots, |\mathcal{J}|\right\}$, is $(E - \mu)$ -nonresonant for every $\mu \in \sigma(H_{\Lambda_L(\mathbf{u}_{\mathcal{J}^c})})$. Otherwise we say $\Lambda_L^{(N)}(\mathbf{u})$ is E -left resonant (or LR).
- (ii) $\Lambda_L^{(N)}(\mathbf{u})$ is E -right nonresonant (or RNR) if and only if for every box $\Lambda_{K_j\ell}(\mathbf{a}) \subseteq \Lambda_L(\mathbf{u}_{\mathcal{J}^c})$ with $\mathbf{a} \in \Xi_{L,\ell}(\mathbf{u}_{\mathcal{J}^c})$ and $j \in \left\{1, 2, \dots, |\mathcal{J}^c|\right\}$ is $(E - \lambda)$ -nonresonant for every $\lambda \in \sigma(H_{\Lambda_L(\mathbf{u}_{\mathcal{J}})})$. Otherwise we say $\Lambda_L^{(N)}(\mathbf{u})$ is E -right resonant (or RR).

- (iii) We say $\Lambda_L^{(N)}(\mathbf{u})$ is E -highly nonresonant (or HNR) if and only if $\Lambda_L^{(N)}(\mathbf{u})$ is E -nonresonant, E -LNR, and E -RNR.

Lemma 3.17. Let $E \in \mathbb{R}$, and $\Lambda_L^{(N)}(\mathbf{u}) = \Lambda_L(\mathbf{u}_{\mathcal{J}}) \times \Lambda_L(\mathbf{u}_{\mathcal{J}^c})$ be a PI N -particle box. Assume that the following are true:

- (i) $\Lambda_L^{(N)}(\mathbf{u})$ is E -HNR.
- (ii) $\Lambda_L^{(N)}(\mathbf{u})$ is E -preregular.

Then $\Lambda_L^{(N)}(\mathbf{u})$ is $(m(L), E)$ -regular, where

$$m(L) \geq m^* - c_1(\ell^{1-\gamma}) - c_2(\ell^{1-\beta}) - c_3\left(\frac{\log L}{L}\right), \quad (3.82)$$

with the constants c_1, c_2, c_3 do not depend on the scale ℓ and $m^* = m_\tau$.

Lemma 3.18. Let $E \in \mathbb{R}$, and $\Lambda_L^{(N)}(\mathbf{u}) = \Lambda_L(\mathbf{u}_{\mathcal{J}}) \times \Lambda_L(\mathbf{u}_{\mathcal{J}^c})$ be a PI N -particle box.

- (i) If $\Lambda_L^{(N)}(\mathbf{u})$ is E -right resonant, then there exists an N -particle box

$$\Lambda = \Lambda_L(\mathbf{u}_{\mathcal{J}}) \times \Lambda_{K_j\ell}(\mathbf{x}), \quad (3.83)$$

where $j \in \{1, 2, \dots, |\mathcal{J}^c|^{\lfloor \mathcal{J}^c \rfloor}\}$, $\mathbf{x} \in \Xi_{L,\ell}(\mathbf{u}_{\mathcal{J}^c})$, and $\Lambda_{K_j\ell}(\mathbf{x}) \subseteq \Lambda_L(\mathbf{u}_{\mathcal{J}^c})$, such that

$$\text{dist}(\sigma(\Lambda), E) < \frac{1}{2}e^{-(K_j\ell)^\beta} \leq \frac{1}{2}e^{-\ell^\beta}. \quad (3.84)$$

- (ii) If $\Lambda_L^{(N)}(\mathbf{u})$ is E -left resonant, then there exists an N -particle box

$$\Lambda = \Lambda_{K_j\ell} \times \Lambda_L(\mathbf{u}_{\mathcal{J}^c}), \quad (3.85)$$

where $j \in \{1, 2, \dots, |\mathcal{J}|^{\lfloor \mathcal{J} \rfloor}\}$, $\mathbf{x} \in \Xi_{L,\ell}(\mathbf{u}_{\mathcal{J}})$, and $\Lambda_{K_j\ell}(\mathbf{x}) \subseteq \Lambda_L(\mathbf{u}_{\mathcal{J}})$, such that

$$\text{dist}(\sigma(\Lambda), E) < \frac{1}{2}e^{-(K_j\ell)^\beta} \leq \frac{1}{2}e^{-\ell^\beta}. \quad (3.86)$$

Proof. Let $E \in \mathbb{R}$ and $\Lambda_L^{(N)}(\mathbf{u}) = \Lambda_L(\mathbf{u}_{\mathcal{J}}) \times \Lambda_L(\mathbf{u}_{\mathcal{J}^c})$ be a PI N -particle box. Assume that $\Lambda_L^{(N)}(\mathbf{u})$ is E -right resonant. Then by definition, we can find $\lambda \in \sigma(H_{\Lambda_L(\mathbf{u}_{\mathcal{J}})})$ and an $(N - |\mathcal{J}|)$ -particle box, $\Lambda_{K_j\ell}(\mathbf{x}) \subseteq \Lambda_L(\mathbf{u}_{\mathcal{J}^c})$, with $\mathbf{x} \in \Xi_{L,\ell}(\mathbf{u}_{\mathcal{J}^c})$ and $j \in \{1, 2, \dots, |\mathcal{J}|^{\lfloor \mathcal{J} \rfloor}\}$, such that $\Lambda_{K_j\ell}(\mathbf{x})$ is $(E - \lambda)$ -resonant, i.e., $\text{dist}(\sigma(H_{\Lambda_{K_j\ell}(\mathbf{x})}), E - \lambda) < \frac{1}{2}e^{-(K_j\ell)^\beta}$. Thus there exists $\eta \in \sigma(H_{\Lambda_{K_j\ell}(\mathbf{x})})$ such that

$$|E - \lambda - \eta| < \frac{1}{2}e^{-(K_j\ell)^\beta}. \quad (3.87)$$

Moreover, $\Lambda_L(\mathbf{u}_{\mathcal{J}}) \times \Lambda_L(\mathbf{u}_{\mathcal{J}^c})$ is PI and $\Lambda_{K_j\ell} \subseteq \Lambda_L(\mathbf{u}_{\mathcal{J}^c})$, so if we take $\Lambda = \Lambda_L(\mathbf{u}_{\mathcal{J}}) \times \Lambda_{K_j\ell}(\mathbf{x})$ we get $H_\Lambda^{(N)} = H_{\Lambda_L(\mathbf{u}_{\mathcal{J}})} \otimes I + I \otimes H_{\Lambda_{K_j\ell}}$, which means that

$$\sigma(H_\Lambda) = \sigma(H_{\Lambda_L(\mathbf{u}_{\mathcal{J}})}) + \sigma(H_{\Lambda_{K_j\ell}}). \quad (3.88)$$

Hence, if a PI N -particle box $\Lambda_L^{(N)}(\mathbf{u}) = \Lambda_L(\mathbf{u}_{\mathcal{J}}) \times \Lambda_L(\mathbf{u}_{\mathcal{J}^c})$ is E -right resonant, then there exists an N -particle box $\Lambda = \Lambda_L(\mathbf{u}_{\mathcal{J}}) \times \Lambda_{K_j\ell}$, where $\Lambda_{K_j\ell} \subseteq \Lambda_L(\mathbf{u}_{\mathcal{J}^c})$, such that

$$\text{dist}(\sigma(H_\Lambda), E) < \frac{1}{2}e^{-(K_j\ell)^\beta}.$$

The same argument applies to a PI N -particle box being E -left resonant. \square

We now state the energy interval multiscale analysis. Given $m > 0$, $L \in \mathbb{N}$, $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}$, and an interval I , we define the event

$$R(m, I, \mathbf{x}, \mathbf{y}, L, N) = \left\{ \exists E \in I \text{ such that } \Lambda_L^{(N)}(\mathbf{x}) \text{ and } \Lambda_L^{(N)}(\mathbf{y}) \text{ are not } (m, E)\text{-regular} \right\}. \quad (3.89)$$

Proposition 3.19. . Let $\zeta, \tau, \beta, \zeta_1, \zeta_2, \gamma$ as in (3.5) and $0 < m_0 < m^*$. Then there exists a length scale Z_3^* such that, given an interval $I \subseteq \mathbb{R}$, if for some $L_0 \geq Z_3^*$ we can verify

$$\mathbb{P}\left\{R(m_0, I, \mathbf{x}, \mathbf{y}, L_0, N)\right\} \leq e^{-L_0^{\zeta_2}} \quad (3.90)$$

for every pair of partially separable N -particle boxes $\Lambda_{L_0}^N(\mathbf{x})$ and $\Lambda_{L_0}^{(N)}(\mathbf{y})$, then, for all $k = 0, 1, 2, \dots$ we have, setting $L_{k+1} = L_k^\gamma = L_0^{\gamma^k}$, that

$$\mathbb{P}\left\{R\left(\frac{m_0}{2}, I, \mathbf{x}, \mathbf{y}, L_k, N\right)\right\} \leq e^{-L_k^{\zeta_2}} \quad (3.91)$$

for every pair of partially separable N -particle boxes $\Lambda_{L_k}^{(N)}(\mathbf{x})$ and $\Lambda_{L_k}^{(N)}(\mathbf{y})$.

Proof. Given ℓ (sufficiently large) and $0 < m_\ell < m^*$, we set $L = \ell^\gamma$ and take m_L as in (3.30). If ℓ is large, we have $m(\ell) > m_\ell$, where $m(\ell)$ is given in (3.82), and conclude that $m(L) \geq m(\ell) > m_\ell > m_L$.

We start by showing that if

$$\mathbb{P}\left\{R(m_\ell, I, \mathbf{x}, \mathbf{y}, \ell, N)\right\} \leq e^{-\ell^{\zeta_2}} \quad (3.92)$$

for every pair of partially separable N -particle boxes $\Lambda_\ell^{(N)}(\mathbf{x})$ and $\Lambda_\ell^{(N)}(\mathbf{y})$, then

$$\mathbb{P}\left\{R(m_L, I, \mathbf{x}, \mathbf{y}, L, N)\right\} \leq e^{-L^{\zeta_2}} \quad (3.93)$$

for every pair of partially separable N -particle boxes $\Lambda_L^{(N)}(\mathbf{x})$ and $\Lambda_L^{(N)}(\mathbf{y})$.

Let $\Lambda_L^{(N)}(\mathbf{x})$ and $\Lambda_L^{(N)}(\mathbf{y})$ be a pair of partially separable N -particle boxes. Let $J \in 2N$. Let \mathcal{B}_J be the event that there exists $E \in I$ such that either $\mathcal{C}_{L,\ell}(\mathbf{x})$ or $\mathcal{C}_{L,\ell}(\mathbf{y})$ contains J pairwise ℓ -distant FI boxes that are (m_ℓ, E) -nonregular, and let \mathcal{A} be the event that there exists $E \in I$ such that either $\mathcal{C}_{L,\ell}(\mathbf{x})$ or $\mathcal{C}_{L,\ell}(\mathbf{y})$ contains one PI box that is not E -preregular. If $\omega \in \mathcal{B}_J^c \cap \mathcal{A}^c$, then for all $E \in I$ the following holds:

- (i) $\mathcal{C}_{L,\ell}(\mathbf{x})$ and $\mathcal{C}_{L,\ell}(\mathbf{y})$ contain at most $J - 1$ pairwise ℓ -distant FI (m_ℓ, E) -nonregular boxes.
- (ii) Every PI box in $\mathcal{C}_{L,\ell}(\mathbf{x})$ and $\mathcal{C}_{L,\ell}(\mathbf{y})$ is E -preregular.

We also define the event

$$\mathcal{U}_J = \bigcup_{\Lambda' \in \mathcal{M}_{\mathbf{x}}, \Lambda'' \in \mathcal{M}_{\mathbf{y}}} \left\{ \text{dist}(\sigma(H_{\Lambda'}), \sigma(H_{\Lambda''})) < e^{-\ell^\beta} \right\}, \quad (3.94)$$

where, given an N -particle box $\Lambda_L^{(N)}(\mathbf{a})$, by $\mathcal{M}_{\mathbf{a}}$ we denote the collection of all boxes of the following three types:

- (i) $\Lambda_L^{(N)}(\mathbf{a})$,
- (ii) $\Lambda_{K_{j_t}\ell}(\mathbf{u}) \subseteq \Lambda_L^{(N)}(\mathbf{a})$, where $\mathbf{u} \in \Xi_{L,\ell}(\mathbf{a})$, $t \in \{1, 2, \dots, T_{\mathbf{a}} \mid T_{\mathbf{a}} \leq JN^N\}$, and $j_t \in \{1, 2, \dots, JN^N\}$,

- (iii) $\Lambda = \Lambda_{t_1}^{\mathcal{J}}(\mathbf{v}) \times \Lambda_L(\mathbf{a}_{\mathcal{J}^c})$, where $\mathbf{v} \in \Xi_{L,\ell}(\mathbf{a}_{\mathcal{J}})$, \mathcal{J} is any non empty subset of $\{1, \dots, N\}$, $t_1 \in \{K_{j_t} \ell \mid t = 1, \dots, T_{\mathbf{v}} \leq |\mathcal{J}|^{|\mathcal{J}|}, j_t \in \{1, 2, \dots, |\mathcal{J}|^{|\mathcal{J}|}\}\}$,

It is clear that $|\mathcal{M}_{\mathbf{a}}| < 2^{N+2} J^2 N^{2N} \left(\frac{2L}{\ell}\right)^{Nd}$, and hence it follows from Corollary 2.4 that

$$\mathbb{P}(\mathcal{U}_{\mathcal{J}}) \leq 2N^{4N+1} J^4 \|\rho\|_{\infty} \left(\frac{2L}{\ell}\right)^{2Nd} L^{Nd} e^{-\ell^{\beta}}. \quad (3.95)$$

Note that for $\omega \in \mathcal{U}_J^c$ and $E \in I$, either every box in $\mathcal{M}_{\mathbf{x}}$ is E -nonresonant or every box in $\mathcal{M}_{\mathbf{y}}$ is E -nonresonant.

Let $\omega \in \mathcal{B}_J^c \cap \mathcal{A}^c \cap \mathcal{U}_J^c$ and $E \in I$. If every box in $\mathcal{M}_{\mathbf{x}}$ is E -nonresonant, then, in particular, every PI box in $\mathcal{C}_{L,\ell}(\mathbf{x})$ is E -HNR and E -preregular, and hence $(m(\ell), E)$ -regular by Lemma 3.17. As $m(\ell) > m_{\ell}$, we conclude that every PI box in $\mathcal{C}_{L,\ell}(\mathbf{x})$ is (m_{ℓ}, E) -regular. Since $\omega \in \mathcal{B}_J^c \cap \mathcal{A}^c$, $\mathcal{C}_{L,\ell}(\mathbf{x})$ contains at most $J - 1$ pairwise ℓ -distant FI (m_{ℓ}, E) -nonregular boxes in $\mathcal{C}_{L,\ell}(\mathbf{x})$, and all other boxes in $\mathcal{C}_{L,\ell}(\mathbf{x})$ are (m_{ℓ}, E) -regular, it follows from Lemma 3.5 that $\Lambda_L^{(N)}(\mathbf{x})$ is (m_L, E) -regular. If there exists a box in $\mathcal{M}_{\mathbf{x}}$ that is E -nonresonant, then every box in $\mathcal{M}_{\mathbf{y}}$ must be E -nonresonant, and thus $\Lambda_L^{(N)}(\mathbf{y})$ is (m_L, E) -regular by the previous argument. Thus for every $E \in I$ either $\Lambda_L^{(N)}(\mathbf{x})$ is (m_L, E) -regular or $\Lambda_L^{(N)}(\mathbf{y})$ is (m_L, E) -regular. It follows that

$$R(m_L, I, \mathbf{x}, \mathbf{y}, L, N) \subseteq (\mathcal{B}_J^c \cap \mathcal{A}^c \cap \mathcal{U}_J^c)^c, \quad (3.96)$$

so

$$\mathbb{P}(R(m_L, I, \mathbf{x}, \mathbf{y}, L, N)) \leq \mathbb{P}(\mathcal{B}_J) + \mathbb{P}(\mathcal{A}) + \mathbb{P}(\mathcal{U}_J). \quad (3.97)$$

Using Lemma 2.10 and Lemma 3.15, we get

$$\mathbb{P}(\mathcal{B}_J) \leq 2 \left(\frac{2L}{\ell}\right)^{2Nd} e^{-\frac{J}{2} L^{\frac{\zeta_2}{\gamma}}} \quad \text{and} \quad \mathbb{P}(\mathcal{A}) \leq 2 \left(\frac{2L}{\ell}\right)^{2Nd} e^{-L^{\frac{\tau}{\gamma}}}. \quad (3.98)$$

We now fix

$$J \in \left(2L^{\beta - \frac{\zeta_2}{\gamma}}, 2L^{\beta - \frac{\zeta_2}{\gamma}} + 2\right] \cap 2\mathbb{N},$$

so (L large) $\mathbb{P}(\mathcal{B}_J) \leq \frac{1}{3} e^{-L^{\zeta_2}}$, $\mathbb{P}(\mathcal{A}) \leq \frac{1}{3} e^{-L^{\zeta_2}}$, and $\mathbb{P}(\mathcal{U}_J) \leq \frac{1}{3} e^{-L^{\zeta_2}}$, and we conclude from (3.97) that

$$\mathbb{P}(R(m_L, I, \mathbf{x}, \mathbf{y}, L, N)) \leq e^{-L^{\zeta_2}}. \quad (3.99)$$

We now take L_0 large enough so that $m(L_0) > m_{L_0} = m_0$, and the above procedure can be carried out with $\ell = L_0$, let $L_{k+1} = L_k^{\gamma}$ for $k = 0, 1, \dots$, and set $m_k = m_{L_k}$. To finish the proof, we just need to make sure $m_{L_k} > \frac{m_0}{2}$ for all $k = 0, 1, \dots$, but this clearly can be achieved by taking L_0 sufficiently large, similarly to the argument in (3.46). \square

Remark 3.20. The proof of Proposition 3.19 gives us more than just our desired conclusion. It shows that for $\omega \in \mathcal{B}_J^c \cap \mathcal{A}^c \cap \mathcal{U}_J^c$ and $E \in I$, either $\Lambda_L^{(N)}(\mathbf{x})$ is (m_L, E) -good or $\Lambda_L^{(N)}(\mathbf{y})$ is (m_L, E) -good. Hence,

$$\mathbb{P}\left\{\exists E \in I \text{ so } \Lambda_{L_k}^{(N)}(\mathbf{x}) \text{ and } \Lambda_{L_k}^{(N)}(\mathbf{y}) \text{ are not } (m_{\zeta_2}, E)\text{-good}\right\} \leq e^{-L_k^{\zeta_2}}. \quad (3.100)$$

As a consequence, we also get a stronger form of Proposition 3.19.

Theorem 3.21. *Let $\zeta_2 \in (\zeta, \tau)$ and $\gamma \in (1, \frac{1}{\zeta_2})$ with $\zeta \gamma^2 < \zeta_2$ be given. Assume there exists a mass $m_{\zeta_2} > 0$, a length scale $L_0 = L_0(\zeta_2)$, and $\delta_{\zeta_2} > 0$ such that if we take $L_{k+1} = L_k^\gamma$, then for every $k \in \mathbb{N}$ and for every $E_1 \in \mathbb{R}$, setting $I(E_1) = [E_1 - \delta_{\zeta_2}, E_1 + \delta_{\zeta_2}]$, we have*

$$\mathbb{P}\left\{\exists E \in I(E_1) \text{ so } \Lambda_{L_k}^{(N)}(\mathbf{a}) \text{ and } \Lambda_{L_k}^{(N)}(\mathbf{b}) \text{ are not } (m_{\zeta_2}, E)\text{-good}\right\} \leq e^{-L_k^{\zeta_2}}, \quad (3.101)$$

for every pair of L_k -distant N -particle boxes $\Lambda_L^{(N)}(\mathbf{a})$ and $\Lambda_L^{(N)}(\mathbf{b})$. Then there exists L_ζ such that for every $E_1 \in \mathbb{R}$ and every $L \geq L_\zeta$

$$\mathbb{P}\left\{\exists E \in I(E_1) \text{ so } \Lambda_L^{(N)}(\mathbf{a}) \text{ and } \Lambda_L^{(N)}(\mathbf{b}) \text{ are not } \left(\frac{m_{\zeta_2}}{2}, E\right)\text{-good}\right\} \leq e^{-L^\zeta},$$

for every pair of L -distant N -particle boxes $\Lambda_L^{(N)}(\mathbf{a})$ and $\Lambda_L^{(N)}(\mathbf{b})$.

Proof. Given a scale L we take K such that $L_K \leq L < L_{K+1}$, and set $\ell = L_{K-1}$. Note that $L_K = \ell^\gamma$ and $L_{K+1} = L_K^\gamma = \ell^{\gamma^2}$, so $L = \ell^{\gamma'}$ with $\gamma \leq \gamma' < \gamma^2$.

Let $\Lambda_L^{(N)}(\mathbf{a})$ and $\Lambda_L^{(N)}(\mathbf{b})$ be a pair of L -distant N -particle boxes. Given $\mathbf{u} \in \Xi_{L,\ell}(\mathbf{a})$ and $\mathbf{v} \in \Xi_{L,\ell}(\mathbf{b})$, we set

$$\mathcal{R}_{\mathbf{u}, \mathbf{v}} = \left\{ \exists E \in I(E_1) \text{ so } \Lambda_\ell^{(N)}(\mathbf{u}) \text{ and } \Lambda_\ell^{(N)}(\mathbf{v}) \text{ are not } (m_{\zeta_2}, E)\text{-good} \right\},$$

and let

$$\mathcal{F}_2 = \bigcup_{\mathbf{u} \in \Xi_{L,\ell}(\mathbf{a}), \mathbf{v} \in \Xi_{L,\ell}(\mathbf{b})} \mathcal{R}_{\mathbf{u}, \mathbf{v}}. \quad (3.102)$$

Let $\omega \in \mathcal{F}_2^c$. Then, either every box in $\mathcal{C}_{L,\ell}^{(N)}(\mathbf{a})$ is (m_{ζ_2}, E) -good, or every box in $\mathcal{C}_{L,\ell}^{(N)}(\mathbf{b})$ is (m_{ζ_2}, E) -good for every $E \in I(E_1)$. Hence, by Lemma 3.10, either $\Lambda_L^{(N)}(\mathbf{a})$ is $(\frac{m_{\zeta_2}}{2}, E)$ -good or $\Lambda_L^{(N)}(\mathbf{b})$ is $(\frac{m_{\zeta_2}}{2}, E)$ -good for every $E \in I(E_1)$ and for every $\omega \in \mathcal{F}_1^c$. The conclusion follows since

$$\mathbb{P}(\mathcal{F}_2) \leq e^{-L^\zeta}. \quad (3.103)$$

□

3.5. Completing the proof of the bootstrap multiscale analysis. Proceeding as in [GK1, Section 6], Theorem 1.4 follows from Propositions 3.2, 3.4, 3.6, plus Proposition 3.8 for Part (i) (the single energy bootstrap multiscale analysis), and Propositions 3.13, 3.19 and 3.21 for Part (ii) (the energy interval bootstrap multiscale analysis).

4. LOCALIZATION FOR THE MULTI-PARTICLE ANDERSON MODEL

In this section we prove Corollary 1.6. We assume that (1.16) holds on an interval I and prove the conclusions of Corollary 1.6 in I . Since $H_\omega^{(N)}$ has compact spectrum, it can be covered by a finite number of intervals where (1.16) holds, and hence the conclusions of Corollary 1.6 hold on \mathbb{R} .

We take L_0 large enough so (1.16) holds for all $L \geq L_0$, and set $L_{k+1} = 2L_k$ for $k \in \mathbb{N}$.

4.1. Anderson Localization.

Proof of Corollary 1.6(i). For $\mathbf{x}_0 \in \mathbb{Z}^{Nd}$ and $k \in \mathbb{N}$, we set

$$A_{k+1}(\mathbf{x}_0) = \left\{ \mathbf{x} \in \mathbb{Z}^{Nd} \mid L_k < d_H(\mathbf{x}, \mathbf{x}_0) \leq L_{k+1} = 2L_k \right\}. \quad (4.1)$$

Then for $\mathbf{x} \in A_{k+1}(\mathbf{x}_0)$, we have $d_H(\mathbf{x}_0, \mathbf{x}) > L_k$. Moreover, it follows from the definition of the Hausdorff distance that

$$d_H(\mathbf{x}, \mathbf{y}) \leq \|\mathbf{x} - \mathbf{y}\| \leq d_H(\mathbf{x}, \mathbf{y}) + \text{diam } \mathbf{x} \quad \text{for } \mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}, \quad (4.2)$$

where

$$\text{diam } \mathbf{x} := \max_{i, j=1, \dots, N} \|x_i - x_j\|. \quad (4.3)$$

Hence $|A_{k+1}(\mathbf{x}_0)| \leq (2L_k + \text{diam } \mathbf{x}_0)^{Nd}$. Let us define the event

$$E_k(\mathbf{x}_0) := \bigcup_{\mathbf{x} \in A_{k+1}(\mathbf{x}_0)} \left\{ \exists E \in I \text{ such that } \Lambda_{L_k}(\mathbf{x}_0) \text{ and } \Lambda_{L_k}(\mathbf{x}) \text{ are } (m, E)\text{-nonregular} \right\}.$$

Applying (1.16), we get

$$\mathbb{P}(E_k(\mathbf{x}_0)) \leq (2L_k + \text{diam } \mathbf{x}_0)^{Nd} e^{-L_k^\zeta}, \quad (4.4)$$

so we have

$$\sum_{k=0}^{\infty} \mathbb{P}(E_k(\mathbf{x}_0)) < \infty. \quad (4.5)$$

It follows from the Borel Cantelli Lemma that

$$\mathbb{P}\{E_k(\mathbf{x}_0) \text{ occurs infinitely often}\} = 0 \quad \text{for all } \mathbf{x}_0 \in \mathbb{Z}^{Nd}, \quad (4.6)$$

so

$$\mathbb{P}\left\{ \bigcup_{\mathbf{x}_0 \in \mathbb{Z}^{Nd}} \{E_k(\mathbf{x}_0) \text{ occurs infinitely often}\} \right\} = 0, \quad (4.7)$$

i.e.,

$$\mathbb{P}\{E_k(\mathbf{x}_0) \text{ occurs infinitely often for some } \mathbf{x}_0 \in \mathbb{Z}^{Nd}\} = 0. \quad (4.8)$$

Let $\Omega_0 := \left\{ E_k(\mathbf{x}_0) \text{ occurs infinitely often for some } \mathbf{x}_0 \in \mathbb{Z}^{Nd} \right\}$. Take $\omega \in \Omega_0$ and let $H = H_\omega$. We will be done if we can prove that every generalized eigenvalue of H in I is actually an eigenvalue by showing that any corresponding generalized eigenfunction has exponential decay.

Let $E \in I$ be a generalized eigenvalue of H with the corresponding nonzero polynomially bounded generalized eigenfunction ψ , that is $H\psi = E\psi$ and for some $C < \infty$, $t \in \mathbb{N}$, we have

$$|\psi(\mathbf{x})| \leq C(1 + \|\mathbf{x}\|)^t \quad \text{for every } \mathbf{x} \in \mathbb{Z}^{Nd}. \quad (4.9)$$

Since ψ is non zero, there exists $\mathbf{x}_0 \in \mathbb{Z}^{Nd}$ such that $\psi(\mathbf{x}_0) \neq 0$. We know that $E_k(\mathbf{x}_0)$ can only occur finitely many times. Thus there exists k_1 such that for every $k > k_1$, and for any $\mathbf{x} \in A_{k+1}(\mathbf{x}_0)$, either $\Lambda_{L_k}(\mathbf{x}_0)$ is (m, E) -regular or $\Lambda_{L_k}(\mathbf{x})$ is (m, E) -regular.

If $E \notin \sigma(H_{\Lambda_{L_k}(\mathbf{x}_0)})$, we have

$$\psi(\mathbf{x}_0) = \sum_{(\mathbf{a}, \mathbf{b}) \in \partial \Lambda_{L_k}(\mathbf{x}_0)} G_{\Lambda_{L_k}(\mathbf{x}_0)}(E; \mathbf{x}_0, \mathbf{a}) \psi(\mathbf{b}). \quad (4.10)$$

Moreover, if $\mathbf{\Lambda}_{L_k}(\mathbf{x}_0)$ is (m, E) -regular, then

- (i) $E \notin \sigma(H_{\mathbf{\Lambda}_{L_k}(\mathbf{x}_0)})$, and
- (ii) $|\psi(\mathbf{x}_0)| \leq s_{Nd} L_k^{Nd-1} e^{-m(\frac{L_k}{2}-1)} |\psi(\mathbf{x}_1)|$ for some $\mathbf{x}_1 \in \partial_+ \mathbf{\Lambda}_{L_k}(\mathbf{x}_0)$, and hence $|\psi(\mathbf{x}_0)| \leq C s_{Nd} L_k^{Nd-1} e^{-m(\frac{L_k}{2}-1)} (1 + \|\mathbf{x}_0\| + L_k)^t$.

Since we know $\psi(\mathbf{x}_0) \neq 0$, this implies there must exist k_2 such that for every $k > k_2$, $\mathbf{\Lambda}_{L_k}(\mathbf{x}_0)$ is not (m, E) -regular. Taking $k_3 = \max\{k_1, k_2\}$, we can conclude that for every $k > k_3$, $\mathbf{\Lambda}_{L_k}(\mathbf{x})$ is (m, E) -regular for every $\mathbf{x} \in A_{k+1}(\mathbf{x}_0)$. For what we are doing, we will be taking k such that

$$L_k \gg \text{diam } \mathbf{x}_0. \quad (4.11)$$

Thus, if $\mathbf{x} \in A_k(\mathbf{x}_0)$ with $k > k_3$, we have $\mathbf{\Lambda}_{L_k}(\mathbf{x})$ is (m, E) -regular and thus

$$\begin{aligned} |\psi(\mathbf{x})| &\leq C s_{Nd} L_k^{Nd-1} e^{-m(\frac{L_k}{2}-1)} (1 + \|\mathbf{x}_0\| + \text{diam } \mathbf{x}_0 + 2L_k)^t \\ &\leq e^{-\frac{m}{4} L_k} \leq e^{-\frac{m}{8} d_H(\mathbf{x}, \mathbf{x}_0)} \leq e^{-\frac{m}{8} (\|\mathbf{x} - \mathbf{x}_0\| - \text{diam } \mathbf{x}_0)} = e^{\frac{m}{8} \text{diam } \mathbf{x}_0} e^{-\frac{m}{8} \|\mathbf{x} - \mathbf{x}_0\|}, \end{aligned} \quad (4.12)$$

provided k is sufficiently large, so ψ decays exponentially. \square

4.2. Dynamical Localization. We will use the generalized eigenfunction expansion for $H_\omega = H_\omega^{(N)}$ to prove dynamical localization (and SUDEC in Subsection 4.3). We will follow the short review (and the notation) given in [GK3, Section 5], and refer to [KKS, Section 3] for full details. We fix $\nu = \frac{Nd+1}{2}$, and for $\mathbf{a} \in \mathbb{Z}^{Nd}$ let $T_{\mathbf{a}}$ denote the operator on $\mathcal{H} = \ell^2(\mathbb{Z}^{Nd})$ given by multiplication by the function $\langle \mathbf{x} - \mathbf{a} \rangle^\nu$, where $\langle \mathbf{x} \rangle = \sqrt{1 + \|\mathbf{x}\|^2}$, and set $T = T_{\mathbf{0}}$. We consider weighted spaces \mathcal{H}_- and \mathcal{H}_+ , operators T_- and T_+ , and spectral measure μ_ω and generalized eigenprojectors $\mathbf{P}_\omega(\lambda)$ in terms of which we have the generalized eigenfunction expansion for the (bounded operator) $H_\omega = H_\omega^{(N)}$ given in [GK3, Eq. (5.23)].

For $\mathbf{x} \in \mathbb{Z}^{Nd}$, we denote $\chi_{\mathbf{x}}$ to be the orthogonal projection onto $\delta_{\mathbf{x}}$ where the family $\{\delta_{\mathbf{x}} \mid \mathbf{x} \in \mathbb{Z}^{Nd}\}$ is the standard orthonormal basis of \mathcal{H} .

Lemma 4.1. *There exists a constant $c = c(d, N) < \infty$ such that for \mathbb{P} almost every ω*

$$\text{tr} \left(T^{-1} f(H_\omega) T^{-1} \right) \leq c \|f\|_\infty. \quad (4.13)$$

Proof. Given $\mathbf{x} \in \mathbb{Z}^{Nd}$, we have

$$\langle \delta_{\mathbf{x}}, T^{-1} f(H_\omega) T^{-1} \delta_{\mathbf{x}} \rangle = \langle \mathbf{x} \rangle^{-2\nu} \langle \delta_{\mathbf{x}}, f(H_\omega) \delta_{\mathbf{x}} \rangle \leq \langle \mathbf{x} \rangle^{-2\nu} \|f\|_\infty. \quad (4.14)$$

It follows that

$$\text{tr} \left(T^{-1} f(H_\omega) T^{-1} \right) \leq \|f\|_\infty \sum_{\mathbf{x} \in \mathbb{Z}^{Nd}} \langle \mathbf{x} \rangle^{-2\nu} = c \|f\|_\infty. \quad (4.15)$$

\square

For $\mathbf{x} \in \mathbb{Z}^{Nd}$, consider $\|T_+ \chi_{\mathbf{x}}\|$ and $\|\chi_{\mathbf{x}} T_-\|$ as operators from \mathcal{H} to \mathcal{H} . Note that

$$\|T_+ \chi_{\mathbf{x}}\| = \|\chi_{\mathbf{x}} T_-\| = \langle \mathbf{x} \rangle^\nu. \quad (4.16)$$

Lemma 4.2. *For \mathbb{P} -almost every ω , for every $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}$, and for μ_ω -almost every λ , we have*

$$\|\chi_{\mathbf{x}} \mathbf{P}_\omega(\lambda) \chi_{\mathbf{y}}\|_1 \leq \langle \mathbf{x} \rangle^\nu \langle \mathbf{y} \rangle^\nu. \quad (4.17)$$

Proof. We have $\|\chi_{\mathbf{x}} \mathbf{P}_{\omega}(\lambda) \chi_{\mathbf{y}}\|_1 = \|\chi_{\mathbf{x}} T_- T_-^{-1} \mathbf{P}_{\omega}(\lambda) T_+^{-1} T_+ \chi_{\mathbf{y}}\|_1$. Thus

$$\|\chi_{\mathbf{x}} \mathbf{P}_{\omega}(\lambda) \chi_{\mathbf{y}}\|_1 \leq \|\chi_{\mathbf{x}} T_-\| \|T_-^{-1} \mathbf{P}_{\omega}(\lambda) T_+^{-1}\|_1 \|T_+ \chi_{\mathbf{y}}\| = \langle \mathbf{x} \rangle^{\nu} \langle \mathbf{y} \rangle^{\nu}, \quad (4.18)$$

since $\|T_-^{-1} \mathbf{P}_{\omega}(\lambda) T_+^{-1}\|_1 = 1$ for μ_{ω} -almost every λ (see [GK3, Eq. (5.23)]). \square

Lemma 4.3. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}$ with $d_H(\mathbf{x}, \mathbf{y}) > \ell$, and suppose*

$$\mathbb{P}\{R(m, I, \mathbf{x}, \mathbf{y}, \ell, N)\} \leq e^{-\ell^{\zeta}}. \quad (4.19)$$

Then for $\omega \notin R(m, I, \mathbf{x}, \mathbf{y}, \ell, N)$ we have

$$\|\chi_{\mathbf{x}} \mathbf{P}_{\omega}(\lambda) \chi_{\mathbf{y}}\|_1 \leq s_{Nd} \ell^{Nd+\nu-1} e^{-m \frac{\ell}{2}} \langle \mathbf{x} \rangle^{\nu} \langle \mathbf{y} \rangle^{\nu} \quad (4.20)$$

for μ_{ω} -almost every $\lambda \in I$.

Proof. Let $\omega \notin R(m, I, \mathbf{x}, \mathbf{y}, \ell, N)$. Then for every $\lambda \in I$, either $\Lambda_{\ell}^{(N)}(\mathbf{x})$ or $\Lambda_{\ell}^{(N)}(\mathbf{y})$ is (m, λ) -regular. Moreover, we have that $\|\chi_{\mathbf{x}} \mathbf{P}_{\omega}(\lambda) \chi_{\mathbf{y}}\|_1 = \|\chi_{\mathbf{y}} \mathbf{P}_{\omega}(\lambda) \chi_{\mathbf{x}}\|_1$, so without loss of generality, we may assume $\Lambda_{\ell}(\mathbf{x})$ is (m, λ) -regular.

For μ_{ω} -almost every $\lambda \in I$, $\psi = \mathbf{P}_{\omega}(\lambda) \chi_{\mathbf{y}} \phi$, with $\phi \in \mathcal{H}$, is a generalized eigenfunction of H_{ω} corresponding to the generalized eigenvalue λ . Then

$$\psi(\mathbf{x}) = \sum_{(\mathbf{a}, \mathbf{b}) \in \partial \Lambda_{\ell}^{(N)}(\mathbf{x})} G_{\Lambda_{\ell}^{(N)}(\mathbf{x})}(\lambda; \mathbf{x}, \mathbf{a}) \psi(\mathbf{b}).$$

Thus it follows from the regularity of $\Lambda_{\ell}(\mathbf{x})$ that

$$\begin{aligned} \|\chi_{\mathbf{x}} \mathbf{P}_{\omega}(\lambda) \chi_{\mathbf{y}}\|_1 &\leq \sum_{(\mathbf{a}, \mathbf{b}) \in \partial \Lambda_{\ell}(\mathbf{x})} \left| G_{\Lambda_{\ell}^{(N)}(\mathbf{x})}(\lambda; \mathbf{x}, \mathbf{a}) \right| \|\chi_{\mathbf{b}} \mathbf{P}_{\omega}(\lambda) \chi_{\mathbf{y}}\|_1 \\ &\leq \left| \partial \Lambda_{\ell}^{(N)}(\mathbf{x}) \right| \max_{(\mathbf{a}, \mathbf{b}) \in \partial \Lambda_{\ell}(\mathbf{x})} \left| G_{\Lambda_{\ell}^{(N)}(\mathbf{x})}(\lambda; \mathbf{x}, \mathbf{a}) \right| \|\chi_{\mathbf{b}} \mathbf{P}_{\omega}(\lambda) \chi_{\mathbf{y}}\|_1 \\ &\leq s_{Nd} \ell^{Nd-1} e^{-m \frac{\ell}{2}} \langle \|\mathbf{x}\| + \frac{\ell}{2} + 1 \rangle^{\nu} \langle \mathbf{y} \rangle^{\nu} \\ &\leq s_{Nd} \ell^{Nd-1} e^{-m \frac{\ell}{2}} 2^{\frac{\nu}{2}} \langle \frac{\ell}{2} + 1 \rangle^{\nu} \langle \mathbf{x} \rangle^{\nu} \langle \mathbf{y} \rangle^{\nu} \\ &\leq s_{Nd} \ell^{Nd+\nu-1} e^{-m \frac{\ell}{2}} \langle \mathbf{x} \rangle^{\nu} \langle \mathbf{y} \rangle^{\nu}, \end{aligned} \quad (4.21)$$

where we used

$$\langle y_1 + y_2 \rangle \leq \sqrt{2} \langle y_1 \rangle \langle y_2 \rangle \quad \text{for } y_1, y_2 \in \mathbb{R}^k, \quad \text{any } k \in \mathbb{N}. \quad (4.22)$$

\square

Corollary 1.6(ii) is an immediate consequence of the following theorem.

Theorem 4.4. *(Decay of the Kernel)*

Let I be an open interval where the conclusions of Theorem 1.4 holds. Then for every $0 < \zeta_4 < 1$ and $\mathbf{y} \in \mathbb{Z}^{Nd}$ there exists a constant $C(\mathbf{y})$ such that

$$\mathbb{E} \left(\sup_{\|g\| \leq 1} \|\chi_{\mathbf{x}} (g \chi_I)(H_{\omega}) \chi_{\mathbf{y}}\|_1 \right) \leq C(\mathbf{y}) e^{-d_H(\mathbf{x}, \mathbf{y})^{\zeta_4}} \quad \text{for all } \mathbf{x} \in \mathbb{Z}^{Nd}, \quad (4.23)$$

where the supremum is taken over all bounded Borel functions g on \mathbb{R} , and $\|g\| = \sup_{t \in \mathbb{R}} |g(t)|$.

Proof. Let us fix $\mathbf{y} \in \mathbb{Z}^{Nd}$. We will apply our main result using $\zeta_2 \in (\zeta, 1)$.

For $\mathbf{x} \in \mathbb{Z}^{Nd}$, let us denote

$$F_{\mathbf{x}}(\omega) = \sup_{\|g\| \leq 1} \|\chi_{\mathbf{x}}(g \chi_I)(H_{\omega}) \chi_{\mathbf{y}}\|_1.$$

Thus our goal is to show that $\mathbb{E}(F_{\mathbf{x}}(\omega)) \leq C e^{-d_H(\mathbf{x}, \mathbf{y})^{\zeta_4}}$ for all $\mathbf{x} \in \mathbb{Z}^d$ for some constant $C = C(\mathbf{y})$.

As in [GK1], we have

$$\begin{aligned} \|\chi_{\mathbf{x}}(g \chi_I)(H_{\omega}) \chi_{\mathbf{y}}\|_1 &\leq \int_I \|\chi_{\mathbf{x}} g(\lambda) \mathbf{P}_{\omega}(\lambda) \chi_{\mathbf{y}}\|_1 d\mu_{\omega}(\lambda) \\ &= \int_I |g(\lambda)| \|\chi_{\mathbf{x}} \mathbf{P}_{\omega}(\lambda) \chi_{\mathbf{y}}\|_1 d\mu_{\omega}(\lambda), \end{aligned} \quad (4.24)$$

and thus

$$F_{\mathbf{x}}(\omega) \leq \int_I \|\chi_{\mathbf{x}} \mathbf{P}_{\omega}(\lambda) \chi_{\mathbf{y}}\|_1 d\mu_{\omega}(\lambda). \quad (4.25)$$

We will divide the proof into the case where $d_H(\mathbf{x}, \mathbf{y}) > L_k$ for some k large enough ($k \geq K_0$), and the case where $d_H(\mathbf{x}, \mathbf{y}) \leq L_{K_0}$.

Case 1: If $d_H(\mathbf{x}, \mathbf{y}) > L_k$ for some $k \geq K_0$, let us take the largest k such that $d_H(\mathbf{x}, \mathbf{y}) > L_k$ but $d_H(\mathbf{x}, \mathbf{y}) \leq L_{k+1}$. Let us denote the set

$$\mathcal{A} = \left\{ \exists E \in I \text{ such that } \mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{x}) \text{ and } \mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{y}) \text{ are } (m, E)\text{-nonregular} \right\}.$$

Then $\mathbb{E}(F_{\mathbf{x}}(\omega)) = \mathbb{E}(F_{\mathbf{x}}(\omega); \omega \in \mathcal{A}) + \mathbb{E}(F_{\mathbf{x}}(\omega); \omega \notin \mathcal{A})$.

To estimate $\mathbb{E}(F_{\mathbf{x}}(\omega); \omega \in \mathcal{A})$, we apply Lemma 4.2 to get

$$\mathbb{E}(F_{\mathbf{x}}(\omega); \omega \in \mathcal{A}) \leq \mathbb{E}(\langle \mathbf{x} \rangle^{\nu} \langle \mathbf{y} \rangle^{\nu} \mu_{\omega}(I); \omega \in \mathcal{A}) = \langle \mathbf{x} \rangle^{\nu} \langle \mathbf{y} \rangle^{\nu} \mathbb{E}(\mu_{\omega}(I) \chi_{\mathcal{A}}(\omega)). \quad (4.26)$$

But we know that $\mathbb{P}(\mathcal{A}) \leq e^{-L_k^{\zeta_2}}$, and

$$\mathbb{E}(\mu_{\omega}(I) \chi_{\mathcal{A}}(\omega)) \leq \mathbb{E}((\mu_{\omega}(I))^2)^{\frac{1}{2}} \mathbb{E}((\chi_{\mathcal{A}}(\omega))^2)^{\frac{1}{2}} = \mathbb{E}((\mu_{\omega}(I))^2)^{\frac{1}{2}} \mathbb{P}(\mathcal{A})^{\frac{1}{2}}, \quad (4.27)$$

so

$$\begin{aligned} \mathbb{E}(F_{\mathbf{x}}(\omega); \omega \in \mathcal{A}) &\leq \langle \mathbf{x} \rangle^{\nu} \langle \mathbf{y} \rangle^{\nu} \mathbb{E}((\mu_{\omega}(I))^2)^{\frac{1}{2}} e^{-\frac{1}{2} L_k^{\zeta_2}} \\ &= C_1 \mathbb{E}((\mu_{\omega}(I))^2)^{\frac{1}{2}} e^{-\frac{1}{2} L_k^{\zeta_2}}, \end{aligned} \quad (4.28)$$

where $C_1 = \langle \mathbf{x} \rangle^{\nu} \langle \mathbf{y} \rangle^{\nu} = C_1(\mathbf{x}, \mathbf{y}, \nu)$.

To estimate $\mathbb{E}(F_{\mathbf{x}}(\omega); \omega \notin \mathcal{A})$, we apply Lemma 4.3 to get

$$\begin{aligned} \mathbb{E}(F_{\mathbf{x}}(\omega); \omega \notin \mathcal{A}) &\leq \mathbb{E}(C_1 s_{Nd} L_k^{Nd-1+\nu} e^{-m \frac{L_k}{2}} \mu_{\omega}(I); \omega \notin \mathcal{A}) \\ &\leq C_1 s_{Nd} L_k^{Nd-1+\nu} e^{-m \frac{L_k}{2}} \mathbb{E}(\mu_{\omega}(I)). \end{aligned} \quad (4.29)$$

Hence

$$\mathbb{E}(F_{\mathbf{x}}(\omega)) \leq C_1 \mathbb{E}((\mu_{\omega}(I))^2)^{\frac{1}{2}} e^{-\frac{1}{2} L_k^{\zeta_2}} + C_1 s_{Nd} L_k^{Nd-1+\nu} e^{-m \frac{L_k}{2}} \mathbb{E}(\mu_{\omega}(I)).$$

Since $k \geq \mathcal{K}_0$, i.e. L_k is large enough, we can conclude

$$\mathbb{E}(F_{\mathbf{x}}(\omega)) \leq 2C_1 \mathbb{E}((\mu_{\omega}(I))^2)^{\frac{1}{2}} e^{-\frac{1}{2}L_k^{\zeta_2}}. \quad (4.30)$$

But $d_H(\mathbf{x}, \mathbf{y}) \leq L_{k+1}$, and $\|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\| \leq L_{k+1} + \text{diam } \mathbf{y} + \|\mathbf{y}\|$, so

$$\langle \mathbf{x} \rangle^{\nu} \leq 2^{\frac{\nu}{2}} \langle L_{k+1} \rangle^{\nu} \langle \text{diam } \mathbf{y} + \|\mathbf{y}\| \rangle^{\nu}, \quad (4.31)$$

which means

$$\mathbb{E}(F_{\mathbf{x}}(\omega)) \leq C_2 e^{-\frac{1}{2}L_k^{\zeta_2}}, \quad (4.32)$$

where $C_2 = 2^{1+\frac{\nu}{2}} \langle L_{k+1} \rangle^{\nu} \langle \text{diam } \mathbf{y} + \|\mathbf{y}\| \rangle^{\nu} \langle \mathbf{y} \rangle^{\nu} \mathbb{E}((\mu_{\omega}(I))^2)^{\frac{1}{2}}$. If we take L_k to be sufficiently large (which is the same as saying \mathcal{K}_0 is sufficiently large),

$$\begin{aligned} \mathbb{E}(F_{\mathbf{x}}(\omega)) &\leq \langle \text{diam } \mathbf{y} + \|\mathbf{y}\| \rangle^{2\nu} e^{-\frac{1}{4}L_k^{\zeta_2}} = \langle \text{diam } \mathbf{y} + \|\mathbf{y}\| \rangle^{2\nu} e^{-\frac{1}{8}L_{k+1}^{\zeta_2}} \\ &\leq \langle \text{diam } \mathbf{y} + \|\mathbf{y}\| \rangle^{2\nu} e^{-L_{k+1}^{\zeta_2}} \leq \langle \text{diam } \mathbf{y} + \|\mathbf{y}\| \rangle^{2\nu} e^{-d_H(\mathbf{x}, \mathbf{y})^{\zeta_2}}. \end{aligned} \quad (4.33)$$

Case 2: If $d_H(\mathbf{x}, \mathbf{y}) \leq L_{\mathcal{K}_0}$, we have $d_H(\mathbf{x}, \mathbf{y}) \leq L_{\mathcal{K}_0}$. Once again we apply Lemma 4.2 to get

$$\begin{aligned} \mathbb{E}(F_{\mathbf{x}}(\omega)) &\leq \mathbb{E}(\langle \mathbf{x} \rangle^{\nu} \langle \mathbf{y} \rangle^{\nu} \mu_{\omega}(I)) = \langle \mathbf{x} \rangle^{\nu} \langle \mathbf{y} \rangle^{\nu} \mathbb{E}(\mu_{\omega}(I)) \\ &\leq \langle d_H(\mathbf{x}, \mathbf{y}) + \text{diam } \mathbf{y} + \|\mathbf{y}\| \rangle^{\nu} \langle \mathbf{y} \rangle^{\nu} \mathbb{E}(\mu_{\omega}(I)) \\ &\leq \langle L_{\mathcal{K}_0} + \text{diam } \mathbf{y} + \|\mathbf{y}\| \rangle^{\nu} \langle \mathbf{y} \rangle^{\nu} \mathbb{E}(\mu_{\omega}(I)) \leq C_2 e^{L_{\mathcal{K}_0}^{\zeta_2}} e^{-d_H(\mathbf{x}, \mathbf{y})^{\zeta_2}}, \end{aligned} \quad (4.34)$$

where $C_2 = \langle L_{\mathcal{K}_0} + \text{diam } \mathbf{y} + \|\mathbf{y}\| \rangle^{\nu} \langle \mathbf{y} \rangle^{\nu} \mathbb{E}(\mu_{\omega}(I))$.

Thus, we get

$$\mathbb{E}(F_{\mathbf{x}}(\omega)) \leq \begin{cases} C_2 e^{L_{\mathcal{K}_0}^{\zeta_2}} e^{-d_H(\mathbf{x}, \mathbf{y})^{\zeta_2}}, & \text{provided } d_H(\mathbf{x}, \mathbf{y}) \leq L_{\mathcal{K}_0} \\ (1 + \text{diam } \mathbf{y} + \|\mathbf{y}\|)^{2\nu} e^{-d_H(\mathbf{x}, \mathbf{y})^{\zeta_2}}, & \text{provided } d_H(\mathbf{x}, \mathbf{y}) > L_{\mathcal{K}_0} \end{cases}. \quad (4.35)$$

To get our desired result, we can just take

$$C = C(y) = \left(1 + L_{\mathcal{K}_0} + \text{diam } \mathbf{y} + \|\mathbf{y}\|\right)^{2\nu} \left(\mathbb{E}(\mu_{\omega}(I)) + 1\right). \quad (4.36)$$

□

4.3. SUDEC. To prove Corollary 1.6(iii) we follow [GK2]. Note that for all $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^{Nd}$ we have $\|\chi_{\mathbf{b}} T_{\mathbf{a}}\| \leq \langle \mathbf{a} - \mathbf{b} \rangle^{\nu}$, and it follows from (4.22) that

$$\|T_{\mathbf{b}}^{-1} T_{\mathbf{a}}\| \leq 2^{\frac{\nu}{2}} \langle \mathbf{b} - \mathbf{a} \rangle^{\nu}. \quad (4.37)$$

We write $E_A(H_{\omega}) := \chi_A(H_{\omega})$ for a Borel measurable set $A \subset \mathbb{R}$, and let $E_{\lambda}(H_{\omega}) := E_{\{\lambda\}}(H_{\omega}) = \chi_{\{\lambda\}}(H_{\omega})$.

Definition 4.5. Given ω , $\lambda \in \mathbb{R}$, and $\mathbf{a} \in \mathbb{Z}^{Nd}$, define

$$\mathbf{W}_{\mathbf{a}, \omega}(\lambda) := \begin{cases} \sup_{\phi \in \mathcal{S}_{\omega, \lambda}} \frac{\|\chi_{\mathbf{a}} \mathbf{P}_{\omega}(\lambda) \phi\|}{\|\mathbf{T}_{\mathbf{a}}^{-1} \mathbf{P}_{\omega}(\lambda) \phi\|}, & \text{if } \mathbf{P}_{\omega}(\lambda) \neq 0, \\ 0, & \text{otherwise,} \end{cases} \quad (4.38)$$

where $\mathcal{S}_{\omega, \lambda} = \{\phi \in \mathcal{H}_+ : \mathbf{P}_{\omega}(\lambda)\phi \neq 0\}$. We also define

$$W_{\mathbf{a}, \omega}(\lambda) := \begin{cases} \sup_{\phi \in \mathcal{T}_{\omega, \lambda}} \frac{\|\chi_{\mathbf{a}} E_{\omega}(\lambda)\phi\|}{\|T_{\mathbf{a}}^{-1} E_{\omega}(\lambda)\phi\|}, & \text{if } E_{\omega}(\lambda) \neq 0, \\ 0, & \text{otherwise,} \end{cases} \quad (4.39)$$

where $\mathcal{T}_{\omega, \lambda} = \{\phi \in \mathcal{H} : E_{\omega}(\lambda)\phi \neq 0\}$, and

$$Z_{\mathbf{a}, \omega}(\lambda) := \begin{cases} \frac{\|\chi_{\mathbf{a}} E_{\omega}(\lambda)\|_2}{\|T_{\mathbf{a}}^{-1} E_{\omega}(\lambda)\|_2}, & \text{if } E_{\omega}(\lambda) \neq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (4.40)$$

Note that $Z_{\mathbf{a}, \omega}(\lambda) \leq W_{\mathbf{a}, \omega}(\lambda) \leq \mathbf{W}_{\mathbf{a}, \omega}(\lambda) \leq 1$ (see [GK2]).

Remark 4.6. Let $\phi \in \mathcal{H}_+$, then $\chi_{\mathbf{a}} \mathbf{P}_{\omega}(\lambda)\phi = \chi_{\mathbf{a}} T_{\mathbf{a}} T_{\mathbf{a}}^{-1} \mathbf{P}_{\omega}(\lambda)\phi$. Then

$$\begin{aligned} \|\chi_{\mathbf{a}} \mathbf{P}_{\omega}(\lambda)\phi\| &= \|\chi_{\mathbf{a}} T_{\mathbf{a}} T_{\mathbf{a}}^{-1} \mathbf{P}_{\omega}(\lambda)\phi\| \leq \|\chi_{\mathbf{a}} T_{\mathbf{a}}\| \|T_{\mathbf{a}}^{-1} \mathbf{P}_{\omega}(\lambda)\phi\| \\ &\leq \|T_{\mathbf{a}}^{-1} \mathbf{P}_{\omega}(\lambda)\phi\|. \end{aligned} \quad (4.41)$$

Thus $\mathbf{W}_{\mathbf{a}, \omega}(\lambda) \leq 1$ for every $\mathbf{a} \in \mathbb{Z}^{Nd}$, every ω , and μ_{ω} -almost every $\lambda \in \mathbb{R}$. Moreover,

$$\begin{aligned} \|T_{\mathbf{a}}^{-1} \mathbf{P}_{\omega}(\lambda)\phi\| &\leq \|T_{\mathbf{a}}^{-1} T\| \|T^{-1} \mathbf{P}_{\omega}(\lambda)\phi\| \leq 2^{\frac{\nu}{2}} \langle \mathbf{a} \rangle^{\nu} \|\mathbf{P}_{\omega}(\lambda)\phi\|_- \\ &\leq 2^{\frac{\nu}{2}} \langle \mathbf{a} \rangle^{\nu} \|\phi\|_+. \end{aligned} \quad (4.42)$$

Remark 4.7. Given $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}$, by [GK2], $\mathbf{W}_{\mathbf{x}, \omega}(\lambda) \mathbf{W}_{\mathbf{y}, \omega}(\lambda)$ is measurable (in λ) with respect to the measure μ_{ω} for \mathbb{P} -a.e. ω . Moreover, we have measurability of $\|\mathbf{W}_{\mathbf{x}, \omega}(\lambda) \mathbf{W}_{\mathbf{y}, \omega}(\lambda)\|_{L^{\infty}(I, d\mu_{\omega}(\lambda))}$ with respect to ω . From Remark 4.6, we also have $\|\mathbf{W}_{\mathbf{x}, \omega}(\lambda) \mathbf{W}_{\mathbf{y}, \omega}(\lambda)\|_{L^{\infty}(I, d\mu_{\omega}(\lambda))} \leq 1$ for \mathbb{P} -a.e. ω .

Lemma 4.8. Let $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}$ and $\omega \in R(m, L, I, \mathbf{x}, \mathbf{y})$, where

$$R(m, L, I, \mathbf{x}, \mathbf{y}) = \left\{ \forall E \in I, \text{ either } \mathbf{\Lambda}_L^{(N)}(\mathbf{x}) \text{ or } \mathbf{\Lambda}_L^{(N)}(\mathbf{y}) \text{ is } (m, E) - \text{regular} \right\}. \quad (4.43)$$

Then there exists a constant $C > 0$ such that

$$\|\mathbf{W}_{\mathbf{x}, \omega}(\lambda) \mathbf{W}_{\mathbf{y}, \omega}(\lambda)\|_{L^{\infty}(I, d\mu_{\omega}(\lambda))} \leq C e^{-m \frac{L}{4}} \quad (4.44)$$

Proof. Let $\omega \in R(m, L, I, \mathbf{x}, \mathbf{y})$. Since $\omega \in R(m, L, I, \mathbf{x}, \mathbf{y})$, we know that for every $\lambda \in I$, either $\mathbf{\Lambda}_L^{(N)}(\mathbf{x})$ or $\mathbf{\Lambda}_L^{(N)}(\mathbf{y})$ is (m, λ) -regular. Without loss of generality, we may assume $\mathbf{\Lambda}_L^{(N)}(\mathbf{x})$ is (m, λ) -regular.

From [KKS] we have that for μ_{ω} -a.e. $\lambda \in I$, $\mathbf{P}(\lambda)\psi := \mathbf{P}_{\omega}(\lambda)\psi$ is a generalized eigenfunction of $H := H_{\omega}$ for every $\phi \in \mathcal{H}_+$. Let $\phi \in \mathcal{H}_+$, and denote $\psi = \mathbf{P}(\lambda)\phi$. Then

$$\begin{aligned} |\chi_{\mathbf{x}} \mathbf{P}(\lambda)\phi| &= |\psi(\mathbf{x})| = \left| \sum_{(\mathbf{a}, \mathbf{b}) \in \delta \mathbf{\Lambda}_L(\mathbf{x})} G_{\mathbf{\Lambda}_{L_k}(\mathbf{x}_0)}(E; \mathbf{x}, \mathbf{a}) \psi(\mathbf{b}) \right| \\ &\leq s_{Nd} L^{Nd-1} e^{-m\|\mathbf{x}-\mathbf{a}\|} |\chi_{\mathbf{b}} \psi| \leq s_{Nd} L^{Nd-1} e^{-m\|\mathbf{x}-\mathbf{a}\|} \|\chi_{\mathbf{b}} T_{\mathbf{x}}\| \|T_{\mathbf{x}}^{-1} \psi\| \\ &\leq s_{Nd} L^{Nd-1} e^{-m\|\mathbf{x}-\mathbf{a}\|} \langle \mathbf{x} - \mathbf{b} \rangle^{\nu} \|T_{\mathbf{x}}^{-1} \mathbf{P}(\lambda)\phi\| \\ &\leq s_{Nd} L^{Nd-1} e^{-m \frac{L}{2}} L^{\nu} \|T_{\mathbf{x}}^{-1} \mathbf{P}(\lambda)\phi\|. \end{aligned} \quad (4.45)$$

Thus there exists $K_0 > 0$ such that if $L \geq K_0$, then

$$|\chi_{\mathbf{x}} \mathbf{P}(\lambda) \phi| \leq e^{-m \frac{L}{4}} \|T_{\mathbf{x}}^{-1} \mathbf{P}(\lambda) \phi\|. \quad (4.46)$$

If $L < K_0$, then there exists a constant $C_1 > 1$ such that

$$\begin{aligned} |\chi_{\mathbf{x}} \mathbf{P}(\lambda) \phi| &\leq s_{Nd} K_0^{Nd-1} e^{-m \frac{L}{2}} K_0^\nu \|T_{\mathbf{x}}^{-1} \mathbf{P}(\lambda) \phi\| \\ &\leq C_1 e^{-m \frac{L}{4}} \|T_{\mathbf{x}}^{-1} \mathbf{P}(\lambda) \phi\|. \end{aligned} \quad (4.47)$$

Using the bound from Remark 4.6 for the term in \mathbf{y} , we get our desired result. \square

Theorem 4.9. *Let I be an open interval where the conclusion of Theorem 1.4 holds. Then for every $\zeta_1 \in (0, 1)$, there exists a constant C_{ζ_1} such that for every $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}$,*

$$\mathbb{E} \left\{ \|\mathbf{W}_{\mathbf{x}, \omega}(\lambda) \mathbf{W}_{\mathbf{y}, \omega}(\lambda)\|_{L^\infty(I, d\mu_\omega(\lambda))} \right\} \leq C_{\zeta_1} e^{-d_H(\mathbf{x}, \mathbf{y})^{\zeta_1}}. \quad (4.48)$$

Proof. Let us denote $f(\omega) = \|\mathbf{W}_{\mathbf{x}, \omega}(\lambda) \mathbf{W}_{\mathbf{y}, \omega}(\lambda)\|_{L^\infty(I, d\mu_\omega(\lambda))}$. Take $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}$. We will divide the proof into several cases.

Case 1: There exists $k \in N$ such that $L_k < d_H(\mathbf{x}, \mathbf{y}) \leq L_{k+1}$; i.e. the pair \mathbf{x} and \mathbf{y} is L_k -distant. Denote $\mathcal{A}_k = R(m, L_k, I, \mathbf{x}, \mathbf{y})$. Then

$$\mathbb{E}\{f(\omega)\} = \mathbb{E}\{f(\omega); \mathcal{A}_k\} + \mathbb{E}\{f(\omega); \mathcal{A}_k^c\} \quad (4.49)$$

On the set \mathcal{A}_k , we have $f(\omega) \leq C e^{-m \frac{L}{4}}$ (Lemma 4.8), so

$$\mathbb{E}\{f(\omega); \mathcal{A}_k\} \leq C e^{-m \frac{L_k}{4}}. \quad (4.50)$$

On the set \mathcal{A}_k^c , we have $f(\omega) \leq 1$ (Remark 4.6); thus

$$\mathbb{E}\{f(\omega); \mathcal{A}_k^c\} \leq \mathbb{P}(\mathcal{A}_k^c) \leq e^{-L_k^\zeta}. \quad (4.51)$$

Hence $\mathbb{E}\{f(\omega)\} \leq C_1 e^{-d_H(\mathbf{x}, \mathbf{y})^{\zeta_1}}$ for a slightly smaller ζ_1 .

Case 2: $d_H(\mathbf{x}, \mathbf{y}) \leq L_0$. By Remark 4.6, we have

$$\begin{aligned} \mathbb{E}\{f(\omega)\} &\leq e^{d_H(\mathbf{x}, \mathbf{y})^{\zeta_1}} e^{-d_H(\mathbf{x}, \mathbf{y})^{\zeta_1}} \leq e^{L_0} e^{-d_H(\mathbf{x}, \mathbf{y})^{\zeta_1}} \\ &= C_2 e^{-d_H(\mathbf{x}, \mathbf{y})^{\zeta_1}}. \end{aligned} \quad (4.52)$$

Thus we get our desired result. \square

The following result of [GK2], though trivial, plays a crucial role in this section, so we state it here without providing the proof.

Lemma 4.10. *Assume that for every $\zeta_1 \in (0, 1)$, we can find a constant C_{ζ_1} such that for every $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}$,*

$$\mathbb{E} \left\{ \|\mathbf{W}_{\mathbf{x}, \omega}(\lambda) \mathbf{W}_{\mathbf{y}, \omega}(\lambda)\|_{L^\infty(I, d\mu_\omega(\lambda))} \right\} \leq C_{\zeta_1} e^{-d_H(\mathbf{x}, \mathbf{y})^{\zeta_1}}.$$

Then for any $\zeta \in (0, 1)$ there exists a constant C_ζ such that

$$\mathbb{E} \left\{ \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}} e^{d_H(\mathbf{x}, \mathbf{y})^\zeta} \langle \mathbf{x} \rangle^{-2\nu} \|\mathbf{W}_{\mathbf{x}, \omega}(\lambda) \mathbf{W}_{\mathbf{y}, \omega}(\lambda)\|_{L^\infty(I, d\mu_\omega(\lambda))} \right\} < C_\zeta.$$

Thus, it follows that for \mathbb{P} -a.e. ω we have

$$\sum_{\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}} e^{d_H(\mathbf{x}, \mathbf{y})^\zeta} \langle \mathbf{x} \rangle^{-2\nu} \|\mathbf{W}_{\mathbf{x}, \omega}(\lambda) \mathbf{W}_{\mathbf{y}, \omega}(\lambda)\|_{L^\infty(I, d\mu_\omega(\lambda))} < \infty.$$

Corollary 4.11. *Suppose that for every $\zeta_1 \in (0, 1)$, we can find a constant C_{ζ_1} such that for every $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}$,*

$$\mathbb{E}\left\{\|\mathbf{W}_{\mathbf{x}, \omega}(\lambda) \mathbf{W}_{\mathbf{y}, \omega}(\lambda)\|_{L^\infty(I, d\mu_\omega(\lambda))}\right\} \leq C_{\zeta_1} e^{-d_H(\mathbf{x}, \mathbf{y})^{\zeta_1}}.$$

Then for \mathbb{P} -a.e. ω , H_ω exhibits pure point spectrum in the interval (a, b) with the corresponding eigenfunctions decaying exponentially fast at infinity. Moreover, for μ_ω -a.e. $\lambda \in I$, λ is an eigenvalue of H_ω with finite multiplicity.

Proof. Since we know that there exists Ω_1 where $\mathbb{P}(\Omega_1) = 1$ and for every $\omega \in \Omega_1$

$$\sum_{\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}} e^{d_H(\mathbf{x}, \mathbf{y})^\zeta} \langle \mathbf{x} \rangle^{-2\nu} \|\mathbf{W}_{\mathbf{x}, \omega}(\lambda) \mathbf{W}_{\mathbf{y}, \omega}(\lambda)\|_{L^\infty(I, d\mu_\omega(\lambda))} < \infty,$$

let us take $\omega \in \Omega_1$. Then there exists a constant $C_\omega = C_{\omega, \zeta}$ such that

$$\sum_{\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}} \langle \mathbf{x} \rangle^{-2\nu} e^{d_H(\mathbf{x}, \mathbf{y})^\zeta} \|\mathbf{W}_{\mathbf{x}, \omega}(\lambda) \mathbf{W}_{\mathbf{y}, \omega}(\lambda)\|_{L^\infty(I, d\mu_\omega(\lambda))} < C_\omega. \quad (4.53)$$

It follows from Lemma 4.10 that for any $\phi \in \mathcal{H}_+$, any $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}$, and for μ_ω -a.e. $\lambda \in I$, we have

$$\|\chi_{\mathbf{x}} \mathbf{P}_\omega(\lambda) \phi\| \|\chi_{\mathbf{y}} \mathbf{P}_\omega(\lambda) \phi\| \leq C_\omega \langle \mathbf{x} \rangle^{2\nu} \|T_{\mathbf{x}}^{-1} \mathbf{P}_\omega(\lambda) \phi\| \|T_{\mathbf{y}}^{-1} \mathbf{P}_\omega(\lambda) \phi\| e^{-d_H(\mathbf{x}, \mathbf{y})^\zeta}. \quad (4.54)$$

But $\|T_{\mathbf{a}}^{-1} \mathbf{P}_\omega(\lambda) \phi\| \leq 2^{\frac{\nu}{2}} \langle \mathbf{a} \rangle^\nu \|\phi\|_+$, so

$$\|\chi_{\mathbf{x}} \mathbf{P}_\omega(\lambda) \phi\| \|\chi_{\mathbf{y}} \mathbf{P}_\omega(\lambda) \phi\| \leq C_\omega 2^\nu \langle \mathbf{x} \rangle^{3\nu} \langle \mathbf{y} \rangle^\nu \|\phi\|_+^2 e^{-d_H(\mathbf{x}, \mathbf{y})^\zeta}. \quad (4.55)$$

We know there exists a constant $\mathcal{Z} < \infty$ such that if $d_H(\mathbf{x}, \mathbf{y}) \geq \mathcal{Z}$ we get $\langle \mathbf{y} \rangle^\nu e^{-d_H(\mathbf{x}, \mathbf{y})^\zeta} \leq e^{-\frac{1}{2}d_H(\mathbf{x}, \mathbf{y})^\zeta}$. Moreover, there exists a constant $\tilde{C} < \infty$ such that if $d_H(\mathbf{x}, \mathbf{y}) \leq \mathcal{Z}$, then $\langle \mathbf{y} \rangle^\nu e^{-d_H(\mathbf{x}, \mathbf{y})^\zeta} \leq \tilde{C}$. So for every $\mathbf{y} \in \mathbb{Z}^{Nd}$,

$$\|\chi_{\mathbf{x}} \mathbf{P}_\omega(\lambda) \phi\| \|\chi_{\mathbf{y}} \mathbf{P}_\omega(\lambda) \phi\| \leq C_1 \langle \mathbf{x} \rangle^{3\nu} e^{-\frac{1}{2}d_H(\mathbf{x}, \mathbf{y})^\zeta} \|\phi\|_+^2 \quad (4.56)$$

for some constant $C_1 = C_1(\omega, \zeta)$. In particular, if $\mathbf{P}_\omega(\lambda) \phi$ is a generalized eigenfunction of H_ω , then we can pick $\mathbf{x}_0 \in \mathbb{Z}^{Nd}$ such that $\|\chi_{\mathbf{x}_0} \mathbf{P}_\omega(\lambda) \phi\| \neq 0$. So we get that for every $\mathbf{y} \in \mathbb{Z}^{Nd}$

$$\|\chi_{\mathbf{y}} \mathbf{P}_\omega(\lambda) \phi\| \leq \frac{C_1 \langle \mathbf{x}_0 \rangle^{3\nu} e^{-\frac{1}{2}d_H(\mathbf{x}_0, \mathbf{y})^\zeta} \|\phi\|_+^2}{\|\chi_{\mathbf{x}_0} \mathbf{P}_\omega(\lambda) \phi\|}. \quad (4.57)$$

It follows that $\mathbf{P}_\omega(\lambda) \phi \in \mathcal{H}$, and hence μ_ω -a.e. $\lambda \in I$ is an eigenvalue of H_ω .

To show finite multiplicity, it is enough for us to show that $\text{tr } E_\lambda(H_\omega) := \text{tr } \chi_{\{\lambda\}}(H_\omega) < \infty$. But

$$\begin{aligned} \mu_\omega(\lambda) \text{tr } E_\lambda(H_\omega) &= \|T^{-1} E_\lambda(H_\omega)\|_2^2 \text{tr } E_\lambda(H_\omega) = \text{tr} \left(E_\lambda(H_\omega) T^{-2} E_\lambda(H_\omega) \right) \text{tr } E_\lambda(H_\omega) \\ &= \text{tr} \left(T^{-2} E_\lambda(H_\omega) \right) \text{tr } E_\lambda(H_\omega) \leq \text{tr} \left(\sum_{\mathbf{x} \in \mathbb{Z}^{Nd}} \langle \mathbf{x} \rangle^{-2\nu} \chi_{\mathbf{x}} E_\lambda(H_\omega) \right) \text{tr} \left(\sum_{\mathbf{y} \in \mathbb{Z}^{Nd}} \chi_{\mathbf{y}} E_\lambda(H_\omega) \right) \\ &\leq \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}} \langle \mathbf{x} \rangle^{-2\nu} \|\chi_{\mathbf{x}} E_\lambda(H_\omega)\|_2^2 \|\chi_{\mathbf{y}} E_\lambda(H_\omega)\|_2^2 \\ &\leq \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}} \langle \mathbf{x} \rangle^{-2\nu} Z_{\mathbf{x}, \omega}(\lambda)^2 \|T_{\mathbf{x}}^{-1} E_\omega(\lambda)\|_2^2 Z_{\mathbf{y}, \omega}(\lambda)^2 \|T_{\mathbf{y}}^{-1} E_\omega(\lambda)\|_2^2. \end{aligned} \quad (4.58)$$

Since $\|T_{\mathbf{x}}^{-1}E_{\omega}(\lambda)\|_2^2$ is bounded uniformly for every $\mathbf{x} \in \mathbb{Z}^{Nd}$, every ω , and every λ , and $Z_{\mathbf{a},\omega}(\lambda) \leq W_{\mathbf{a},\omega}(\lambda) \leq \mathbf{W}_{\mathbf{a},\omega}(\lambda) \leq 1$, we get

$$\begin{aligned} \mu_{\omega}(\lambda) \operatorname{tr} E_{\lambda}(H_{\omega}) &\leq C_2^4 \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}} \langle \mathbf{x} \rangle^{-2\nu} Z_{\mathbf{x},\omega}(\lambda)^2 Z_{\mathbf{y},\omega}(\lambda)^2 \\ &\leq C_2^4 \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}} \langle \mathbf{x} \rangle^{-2\nu} Z_{\mathbf{x},\omega}(\lambda) Z_{\mathbf{y},\omega}(\lambda). \end{aligned} \quad (4.59)$$

The result now follows from Lemma 4.8. \square

Proof of Corollary 1.6 (iii). Let us take $\zeta_1 > \zeta$, and let $E_{n,\omega}$ be an eigenvalue of H_{ω} . For $\psi, \phi \in \operatorname{Ran} \chi_{E_{n,\omega}}(H_{\omega})$, and $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}$, we have

$$\begin{aligned} \|\chi_{\mathbf{x}}\phi\| \|\chi_{\mathbf{y}}\psi\| &\leq \left(W_{\mathbf{x},\omega}(E_{n,\omega}) W_{\mathbf{y},\omega}(E_{n,\omega}) \right) \left(\|T_{\mathbf{x}}^{-1}\phi\| \|T_{\mathbf{y}}^{-1}\psi\| \right) \\ &\leq \left(\mathbf{W}_{\mathbf{x},\omega}(E_{n,\omega}) \mathbf{W}_{\mathbf{y},\omega}(E_{n,\omega}) \right) \left(\|T_{\mathbf{x}}^{-1}\phi\| \|T_{\mathbf{y}}^{-1}\psi\| \right) \\ &\leq \|\mathbf{W}_{\mathbf{x},\omega}(\lambda) \mathbf{W}_{\mathbf{y},\omega}(\lambda)\|_{L^{\infty}(I, d\mu_{\omega}(\lambda))} \left(\|T_{\mathbf{x}}^{-1}\phi\| \|T_{\mathbf{y}}^{-1}\psi\| \right). \end{aligned} \quad (4.60)$$

Thus, we can apply Lemma 4.10 to get

$$\|\chi_{\mathbf{x}}\phi\| \|\chi_{\mathbf{y}}\psi\| \leq C_{\zeta_1} e^{-d_H(\mathbf{x}, \mathbf{y})^{\zeta}} \langle \mathbf{x} \rangle^{2\nu} \left(\|T_{\mathbf{x}}^{-1}\phi\| \|T_{\mathbf{y}}^{-1}\psi\| \right), \quad (4.61)$$

so applying equation (4.37) we get our desired result. \square

REFERENCES

- [ASFH] Aizenman, M., Schenker, J., Friedrich, R., Hundertmark, D.: Finite volume fractional-moment criteria for Anderson localization. *Comm. Math. Phys.* **224**, 219-253 (2001)
- [AW] Aizenman, M., Warzel, S.: Localization bounds for multiparticle systems. *Commun. Math. Phys.* **290**, 903-934 (2009)
- [CBS] V. Chulaevsky, A. Boutet de Monvel, Y. Suhov: Dynamical localization for a multiparticle model with an alloy-type external random potential. *Nonlinearity*, **24**, 1451-1472 (2011)
- [CS1] V. Chulaevsky, Y. Suhov: Wegner bounds for a two particle tight binding model. *Commun. Math. Phys.* **283**, 479-489 (2008)
- [CS2] V. Chulaevsky, Y. Suhov: Eigenfunctions in a two-particle Anderson tight binding model. *Commun. Math. Phys.* **289**, 701-723 (2009)
- [CS3] V. Chulaevsky, Y. Suhov: Multi-particle Anderson Localization: Induction on the number of particles. *Math. Phys. Anal. Geom.*, no. 2, 117-139 (2009)
- [DK] von Dreifus, H., Klein, A.: A new proof of localization in the Anderson tight binding model. *Comm. Math. Phys.* **124**, 285-299 (1989).
- [GK1] Germinet, F., Klein, A.: Bootstrap multiscale Analysis and Localization in Random Media. *Commun. Math. Phys.* **222**, 415-448 (2001)
- [GK2] Germinet, F., Klein, A.: New Characterization of the region of complete localization for random Schrödinger operators. *J. Stats. Phys.* **122**, 73-94 (2006)
- [GK3] Germinet, F., Klein, A.: A comprehensive proof of localization for continuous Anderson models with singular random potentials. *J. Eur. Math. Soc.* **15**, 53-143 (2013)
- [KKS] Klein, A., Koines, A., Seifert, M.: Generalized eigenfunctions for waves in inhomogeneous media. *J. Funct. Anal.* **190**, 255-291 (2002)
- [Ki] Kirsch, W.: An invitation to random Schrödinger operators. In *Random Schrödinger Operators. Panoramas et Syntheses* **25**. Societe Mathematique de France, Paris, 1-119 (2008)
- [Kl] Klein, A.: multiscale analysis and localization of random operators. In *Random Schrödinger Operators. Panoramas et Syntheses* **25**. Societe Mathematique de France, Paris, 121-159 (2008)

UNIVERSITY OF CALIFORNIA, IRVINE, DEPARTMENT OF MATHEMATICS, IRVINE, CA 92697-3875,
USA

E-mail address, A. Klein: `aklein@uci.edu`

E-mail address, S.T. Nguyen: `sondgnguyen1@gmail.com`